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GROUNDING, DEPENDENCE, AND PARADOX*

1. INTRODUCTION

The idea that grounding is an important component of our intuitive notion of well-definedness has long formed part of the conceptual background of axiomatic set theory.[1] Yet only in recent years have we come to appreciate how heavily grounding figures in our intuitions of *semantical* well-definedness. The culmination of this developing appreciation, for the present at least, is Kripke's celebrated Theory of Truth; and while it would be a mistake to say that Kripke's ideas came as a complete surprise to concerned semanticists, it would be a rather small one. After all, here we are, six years later, and while the journals are crammed with informed discussions of Kripke's work on modal logic, intuitionistic logic, designation, and essentialism, they maintain a near unbroken silence on the subject of his Theory of Truth. It may be that this is because Kripke said all there was to say, but more likely the idea is just taking a while to sink in.[2] The first purpose of this paper is to hasten the process by attempting to place some aspects of Kripke's work into formal and philosophical perspective.

The second purpose has to do with my feeling that Kripke has only done half, albeit the first and therefore by far the most important half, of the job. The intuition of grounding is, I want to maintain, a two-sided intuition. On the one side is what I'll call the *inheritance* aspect. This is the aspect lying behind the attempt to understand semantic grounding through the example of a person being taught how to use the word 'true'. Certain basic sentences, not themselves containing 'true', he recognizes as being in accord or disaccord with the facts, and these he is taught to call 'true' and 'false', respectively. Then he learns that the result of suffixing the name of a true or false sentence with the words 'is true' is another sentence of the same kind, along with various other ways in which complex sentences can inherit truth-values from simpler ones. As he generates new truths and falsehoods, he generates new grist for the truth- and falsehood-generation mill, so the process is self-perpetuating and over time our student learns how to assign truth-values

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to more and more statements involving the notion of truth itself. There is no reason to suppose that all statements involving 'true' will become decided in this way, but most will. Indeed, our suggestion is that the "grounded" sentences can be characterized as those which eventually get a truth-value in this process (Kripke, 1975, p. 701).

What distinguishes this way of understanding grounding is that it (so to speak) starts at the bottom and works up, and by so doing invites one to view grounding as a sort of *inheritance* passed along from generation to generation, which only the grounded sentences are fated to receive.[3]

On the other side is what I'll call the *dependence* aspect of our grounding intuition. If the inheritance aspect is the one lying behind the attempt to picture grounding in terms of the *learning* of 'true', then the dependence aspect is the one behind the attempt to picture grounding in terms of the *understanding* of 'true'. What do we do when we have to evaluate a sentence – say "The sentence 'Snow isn't white' is true or the sentence 'The sentence 'Snow is white' is true' is not true" – involving complicated attributions of truth? Evidently, we try to figure out what its truth-value *depends* on, and then what *that* depends on, and so on and so forth in the hope of eventually making our way down to sentences not containing 'true' which can be evaluated by conventional means.

In general, if a sentence asserts that (all, some, most, etc.) of the sentences of a certain class *C* are true, its truth-value can be ascertained if the truth-values of the sentences in the class *C* can be ascertained. If some of these sentences involve the notion of truth, their truth-value in turn must be ascertained by looking at other sentences, and so on. It ultimately this process terminates in sentences not mentioning the concept of truth, so that the truth-value of the original sentence can be ascertained, we call the original sentence grounded; otherwise, ungrounded (Kripke, 1975, pp. 693–4).

What characterizes *this* way of understanding grounding is that it (so to speak) starts at the top and works down, and therefore invites one to view grounding as something for which a sentence *depends* on other sentences.[4]

Now Kripke's Theory is very instructive about the *inheritance* aspect of semantic grounding, but it really does little to supply the *dependence* aspect of our intuition. Not that this is much of a mark against it, for inheritance and dependence are, after all, two sides of the same coin, and to have a theory of either is very nearly as good as to have a theory of both. But the fact that the majority of those who grappled with grounding before Kripke tended to see things from the standpoint of *dependence*[5] suggests that there is something intuitively satisfying about the dependence approach. And from a theoretical point of view, dependence has the advantage of allowing us a little insight into the "fine structure" of semantic grounding (see Sections 8, 9, and 10 for more on this). Hence the paper's second purpose: to give the beginnings of an account of the dependence aspect of grounding.

This paper is divided into two parts, one theoretical and one (comparatively) applied. Sections 2–7 deal with the development of dependence in an abstract setting. Our main result is that any collection with an inheritance-style characterization admits a canonically related dependence-style characterization. In Sections 8–10 we show in a series of applications how the dependence way of doing things can improve our understanding of truth, semantic level, and paradoxicality.

2. INDUCTIVE SPACES AND INHERITANCE[6]

We begin with an arbitrary set U called the **universe**; the members of U are referred to as **objects**. Let P be a collection of subsets of U such that (i) the empty set $\emptyset$ is in P, and (ii) the union of any increasing sequence of members of P is also in P. We call J a **jump operator** on $\langle U, P\rangle$ if J is a monotone operator on P, i.e., if J is a function from P into P, and for any S, S' in $P, S \subseteq S'$ implies $J(S) \subseteq J(S')$. If J is a jump operator on $\langle U, P \rangle$ then $\langle U, P, J \rangle$ is said to be an **inductive space**. Inductive spaces are the object of our study.

Let $\langle U, P, J\rangle$ be an arbitrary inductive space, to be held fixed from this point on, and let S be a member of P. S is said to be **sound** if $S \subseteq J(S)$. If S is sound , we define the sequence $\langle J^\alpha(S) \mid \alpha \text{ an ordinal}\rangle$ – or $\langle J^\alpha(S)\rangle$ for short – by transfinite recursion as follows:

$$J^0(S) = S \text{ and } J^\alpha(S) = J(J^{\alpha-1}(S)) \text{ for all } \alpha > 0,$$

where we agree that '$J^{\alpha-1}(S)$' is to be understood as $\cup_{\beta<\alpha} J^\beta(S)$ if α is a limit ordinal. Our first Proposition shows how the soundness of S guarantees a certain amount of good behaviour on the part of the sequence $\langle J^\alpha(S)\rangle$ it generates.

PROPOSITION 1. Let S be a sound subset of U. Then the sequence $\langle J^\alpha(S)\rangle$ is nondecreasing.

Proof: It suffices to show that for each $\alpha > 0, J^{\alpha-1}(S) \subseteq J^\alpha(S)$. Proof is by transfinite induction. If $\alpha = 1$, then to say that $J^{\alpha-1}(S) \subseteq J^\alpha(S)$ is just to say that S is sound. If α is a successor ordinal, then by hypothesis of induction $J^{\alpha-2}(S) \subseteq J^{\alpha-1}(S)$, whence by monotonicity of J, $J^{\alpha-1}(S) \subseteq J^{\alpha}(S)$. If α is a limit ordinal, then $J^{\alpha-1}(S) = \cup_{\beta<\alpha} J^{\beta}(S)$, so for all $\beta < \alpha$, $J^{\beta}(S) \subseteq J^{\alpha-1}(S)$. Since P is closed under increasing unions, $J^{\alpha-1}(S)$ is in J's domain; applying J to both sides, then, we see that for all $\beta < \alpha$, $J^{\beta+1}(S) \subseteq J^{\alpha}(S)$. Taking the union over all $\beta < \alpha$ of the left hand side gives the desired result that $J^{\alpha-1}(S) \subseteq J^{\alpha}(S)$.

So $\langle J^{\alpha}(S)\rangle$ is nondecreasing for all sound S; the next Proposition shows that on the same assumption $\langle J^{\alpha}(S)\rangle$ can be decomposed into a strictly increasing initial segment and a constant tail.

PROPOSITION 2. Let S be a sound subset of U. Then there is a unique ordinal ρ such that (i) for all $\alpha < \beta \leqslant \rho$, $J^{\alpha}(S) \subsetneqq J^{\beta}(S)$, and (ii) for all $\gamma \geqslant \rho$, $J^{\gamma}(S) = J^{\rho}(S)$.

Proof: I claim first that there is an ordinal κ such that $J(J^{\kappa}(S)) = J^{\kappa}(S)$. Suppose towards a contradiction that there is no such κ. Let μ be the cardinality of U, and for each $\gamma < \mu^+$ (μ^+ is the least cardinal bigger than μ) choose x_γ from $J^{\gamma+1}(S) - J^{\gamma}(S)$. Then $\{x_\gamma \mid \gamma < \mu^+\}$ is a μ^+–membered subset of U, contradicting our choice of μ as U's cardinality. So there are ordi-ordinals κ such that $J(J^{\kappa}(S)) = J^{\kappa}(S)$. Let ρ be the least such. It's a simple exercise in transfinite induction to show that for all $\gamma \geqslant \rho$, $J^{\gamma}(S) = J^{\rho}(S)$, so it remains only to show that if $\alpha < \beta \leqslant \rho$, $J^{\alpha}(S) \neq J^{\beta}(S)$. We know from Proposition 1 that if $J^{\alpha}(S) = J^{\beta}(S)$ for some $\alpha < \beta \leqslant \rho$, then for all γ between α and β, $J^{\alpha}(S) = J^{\gamma}(S) = J^{\beta}(S)$. In particular, $J^{\alpha+1}(S) = J^{\alpha}(S)$, contradicting our choice of ρ as the least ordinal for which this was so.

Thus if S is sound then the sequence $\langle J^{\alpha}(S)\rangle$ strictly increases until it reaches a certain constant value from which it never again deviates. This constant value is denoted by 'S^*', and is called the **closure of S** or the **fixed-point generated by S.** The members of S^* are said to be **grounded relative to S.** Finally, a useful measure of the inductive "height" of a member of S^* relative to S is provided by the **S-level** function L_S, defined by:

$L_S(x)$ is the least α such that x is in $J^{\alpha}(S)$.

Intuitively speaking, every inductive space is endowed with two related but interestingly different structures. On the one hand there is the inheritance structure, and this is what we've been looking at in the present section. On the other is the dependence structure, whose elucidation will be the business of Sections 3–7. Results so far obtained include some which establish relations between the inheritance and dependence structures of inductive spaces and some which deal with properties intrinsic to dependence structures. In most of this paper we'll be looking at only the simplest results of the first kind.

3. JUMP-ENTAILMENT AND SUFFICIENCY SETS

If S is a subset of U such that x is in $J(S)$, then we say that S **jump-entails** or **is sufficient for** x, written $S \mathrm{F}_{J\models} x$.[7] For each x in U we define x's **sufficiency set** $\mathbb{S}(x)$ to be the set of subsets of U which are sufficient for x.

PROPOSITION 3. Sufficiency sets contain all supersets, in P, of their members.

Proof: Let $S \in \mathbb{S}(x)$ and $S \subseteq S' \in P$. Then $S \in \mathbb{S}(x) \Rightarrow S \mathrm{F}_{J\models} x \Rightarrow x \in J(S)$ $\Rightarrow x \in J(S')$ by monotonicity of $J \Rightarrow S' \mathrm{F}_{J\models} x \Rightarrow S' \in \mathbb{S}(x)$.

Of special importance (see Proposition 6 below) are those members of U whose sufficiency sets contain *all* subsets of U in P, i.e., whose sufficiency sets are identical to P. Complete information on these objects is obtained in Proposition 4.

PROPOSITION 4. For all x in U, $\mathbb{S}(x) = P$ if and only if $x \in J(\emptyset)$.

Proof: [$\Leftarrow$] $x \in J(\emptyset) \Rightarrow \emptyset \in \mathbb{S}(x) \Rightarrow \mathbb{S}(x) = P$ by Proposition 3.

[$\Rightarrow$] $\mathbb{S}(x) = P \Rightarrow \emptyset \in \mathbb{S}(x) \Rightarrow \emptyset \mathrm{F}_{J\models} x \Rightarrow x \in J(\emptyset)$.

4. DEPENDENCE RELATIONS

Let S be a subset of U. We define the set $D(S)$ of **S-dependence relations** as follows:

DEFINITION 5. $\Delta \in D(S)$ if and only if Δ is a binary relation on U such that

(α) if $x \in S$ then x bears Δ to nothing;

(β) if $x \notin S$ and $\mathbb{S}(x)$ is non-empty, then for some $Z \in \mathbb{S}(x)$ x bears Δ to exactly the members of Z; and

(γ) if $x \notin S$ and $\mathbb{S}(x)$ is empty, then x bears Δ to itself.[8]

An S-dependence relation recognizes three kinds of objects, according to clauses (α), (β), and (γ) of Definition 5. Either an object x is pre-grounded (i.e., x is in S), or it is provisionally groundable (i.e., x is not in S but there are sets sufficient for x), or it is not even provisionally groundable (i.e., x is not in S and no set is sufficient for x). One can deduce from the observation that clauses (α) and (γ) are categorical – that is, they completely fix the behaviour of Δ towards pre-grounded and non-provisionally-groundable objects – that the S-dependence relations differ among themselves in nothing but the ways in which they attempt to ground the provisionally groundable objects. In fact, each S-dependence relation can be thought of as a network of interdependent gambles, one for each provisionally groundable object, on the question: which way leads down to the ground? The failure of any *particular* member of $D(S)$ to "pay off" in the sense of leading from x to the ground does nothing to show that x is not grounded, for there may be some *other* S-dependence relation that can do the trick. (Compare this to the situation with logical proof, where the failure of any particular attempt to reason backwards from conclusion to premises leaves intact the possibility that the premises imply the conclusion.) Grounding is therefore not so much like covering all the bases as like having a leg to stand on; or, in less figurative language, if *any* S-dependence relation leads from x to the ground, then x is grounded in S.

The following definitions are intended to make these ideas precise. A finite or infinite sequence of objects is called a **Δ-path** if (i) the first of any two consecutive entries bears Δ to the second, and (ii) every entry has only finitely many predecessors.[9] An object is **Δ-grounded** if it heads no infinite Δ-paths. Finally, an object is **grounded in S** if it is Δ-grounded for at least one S-dependence relation Δ. We use 'S_*' to stand for the set of all objects grounded in S.

The notion of grounding *in S* is offered as a sharpening of the *dependence* aspect of our intuition of grounding, just as the notion of grounding *relative to S* (see Section 3) sharpens the inheritance aspect of that intuition. Our main goal is to show that extensionally speaking, the two notions coincide; or, more simply, that $S_* = S^*$.

5. DEPENDENCE RELATIONS AGAIN

In the last section we spoke in an informal way about the "ground" of the inductive space $\langle U, P, J\rangle$ relative to a subset S of U. Formally, the **ground**

over S is defined as the set of all objects x such that for some S-dependence relation Δ, x does not bear Δ to any object. The next result gives a simple and useful characterization of the ground over S in terms of J and S.

PROPOSITION 6. The ground over S is $S \cup J(\emptyset)$.

Proof: [$\subseteq$] Suppose that x is in the ground over S. Then for some Δ in $D(S)$ x bears Δ to nothing. If x is in S then the proposition is satisfied. If not, then either there are or there aren't sets sufficient for x. If there aren't, then x bears Δ to itself by Definition 5(γ), contradicting our choice of Δ as a dependence relation which x doesn't bear to anything. So x's sufficiency set is non-empty. By Definition 5(β) x bears Δ to exactly the members of some set sufficient for x, whence $\emptyset$ is sufficient for x, and x is in $J(\emptyset)$.

[$\supseteq$] Let x be in $S \cup J(\emptyset)$. If x is in S then Definition 5(α) ensures that for *all* Δ in $D(S)$, x bears Δ to nothing. Since $D(S)$ is non-empty, x is in the ground over S. If x is in $J(\emptyset)$ then $\emptyset$ is sufficient for x, whence by Definition 5(β) it is a simple matter to construct a Δ in $D(S)$ such that x bears Δ to nothing. Therefore x is in the ground over S.

Let S be a sound subset of U. Among the members of $D(S)$ there is one, to be referred to as 'Δ_S', of particular importance relative to our goal of proving that $S_* = S^*$. It is defined thus. To begin, Δ_S has to be a binary relation on U such that (α) every element of S bears Δ_S to nothing, and (γ) every object not in S with an empty sufficiency set bears Δ_S to itself only; this much is inevitable from Definition 5. It only remains to set out how Δ_S behaves towards objects not in S with non-empty sufficiency sets, and even here only to specify which of the sets sufficient for x is to be the Z of Definition 5(β). There are two cases: x is in S^* or x isn't in S^*. If the latter, then x isn't going to be Δ_S-grounded anyway, and out of indifference to its precise fate we select an arbitrary Z from $\mathbb{S}(x)$ and let x bear Δ_S to all and only its members. If x is in S^*, then x has an S-level α, equal to the least β such that x is in $J^\beta(S)$, and we simply let x bear Δ_S to all and only the members of $J^{\alpha-1}(S)$. (Recall our convention of letting '$J^{\alpha-1}(S)$' stand for $\cup_{\beta<\alpha} J^\beta(S)$ when α is a limit ordinal.) To see that $J^{\alpha-1}(S)$ is in $\mathbb{S}$ (x), consider that $x \in J^\alpha(S) \Rightarrow x \in J(J^{\alpha-1}(S)) \Rightarrow J^{\alpha-1}(S)$ $\mathrm{F}_{\mathcal{J}\models}$ $x \Rightarrow J^{\alpha-1}(S) \in \mathbb{S}(x)$. Thus clause ($\beta$) is satisfied, and the definition of Δ_S is complete.

The importance of Δ_S derives mainly from its role in the following Proposition, which serves as an essential lemma to our main result below.

PROPOSITION 7. Let S be a sound subset of U. Then if x is in S^*, x is Δ_S-grounded.

Proof: Proof is by transfinite induction on the S-level of x. If $L_S(x) = 0$, then $x \in S$, whence by Definition 5(α) x bears Δ_S to nothing and is therefore Δ_S-grounded. Suppose $L_S(x) = \alpha > 0$. Then x bears Δ_S to exactly the members of $J^{\alpha-1}(S)$. By hypothesis of induction each member of $J^{\alpha-1}(S)$ is Δ_S-grounded, and it follows at once that x is Δ_S-grounded as well.

Now we have one direction of our main result, for it follows from Proposition 7 that $S^* \subseteq S_*$. The other direction will require an additional method to be introduced in the next section, and this same method will incidentally enable us to carry out a dependence-style recovery of the S-levels of all objects in S^*.

6. DEPENDENCE TREES AND RANKS

Dependence Trees: Given x in U and Δ in $D(S)$, we can construct a **dependence tree** $\mathscr{T}(x, \Delta)$ that represents in a graphic way the details of x's grounding or non-grounding, via Δ, in S. The idea is very simple. We begin with an occurrence of x at the top; then we extend lines downward from x to occurrences of each y to which x bears Δ, and similarly for each of these y, and so on and so forth for as long as it takes. Formally it can be done like this. We associate with each y in U an infinite stock $\{y^j\}$ of **occurrences** of y, with j ranging over some initial segment of the ordinals. (Never mind about the exact number of these occurrences – we just require that there be enough to carry us through the construction.) Then $\mathscr{T}(x, \Delta)$ – or $\mathscr{T}$ for short – can be taken to be anything satisfying the following definition:

DEFINITION 8. $\mathscr{T}$ is a binary relation on a set N of occurrences of elements of U such that

(a) x^0 is in N;

(b) there is no n in N bearing $\mathscr{T}$ to x^0;

(c) if y^j is in N then y^j bears $\mathscr{T}$ to exactly one occurrence z^k of each z to which y bears Δ;

(d) if m and n are distinct members of N then there is no p in N such that both m and n bear $\mathscr{T}$ to p;

(e) $N = \{n \mid \text{there is a } \mathscr{T}\text{-path from } x^0 \text{ to } n\}$.

The reader should satisfy him or herself that $\mathscr{T}$, thus defined, really has the intuitive properties outlined above.

Some helpful definitions: N is the set of **nodes** of $\mathscr{T}$. Node m **lies directly above** node n if m bears $\mathscr{T}$ to n. A node is **terminal** if it does not lie directly above any other node. A **branch** is any $\mathscr{T}$-path with first sequent x^0. A branch **terminates** if it is finite in length, and a tree **terminates** if all its branches do.

Since the branches of $\mathscr{T}(x, \Delta)$ correspond in an obvious way with the x-headed Δ-paths, we see that x is Δ-grounded if and only if $\mathscr{T}(x, \Delta)$ terminates. Extensive use of this fact is made in Section 7.

Ranks: If $\mathscr{T}$ is a terminating tree, then there exist standard ways of carrying out definitions and proofs over the nodes of $\mathscr{T}$. *Tree recursion* works like this: if a function is defined on $\mathscr{T}$'s terminal nodes, and if its definition on any given non-terminal node follows from its definition on the nodes lying directly below the given node, then the function is defined on all of $\mathscr{T}$'s nodes. *Tree induction* is similar: if a hypothesis holds of the terminal nodes, and of a non-terminal node provided it holds of the nodes lying directly beneath it, then it holds of all $\mathscr{T}$'s nodes.

To each node n of each terminating dependence tree $\mathscr{T}$ we assign a **$\mathscr{T}$-rank** $R^{\mathscr{T}}(n)$. $\mathscr{T}$-rank assignment is by tree recursion.

DEFINITION 9.

(a) If n is a terminal node and an occurrence of a member of S, then $R^{\mathscr{T}}(n) = 0$.

(b) If n is a terminal node but not an occurrence of a member of S, then $R^{\mathscr{T}}(n) = 1$.

(c) If n is a non-terminal node lying directly above the members of M, then $R^{\mathscr{T}}(n) = \sup \{R^{\mathscr{T}}(m) + 1 \mid m \in M\}$.

Next we assign **ranks** to all terminating trees.

DEFINITION 10. If $\mathscr{T}$ is a terminating tree, then $R(\mathscr{T})$ is the $\mathscr{T}$-rank of $\mathscr{T}$'s topmost node.

Now we're in a position to associate an **S-rank** with each member of S_*. Once again (see the definition of 'S-level' in Section 2) our intention is to provide a measure of the inductive "height" of x with respect to S, only this time as judged from above.

DEFINITION 11. If x is in S_*, then $R_S(x) = \inf\{R(\mathcal{T}(x, \Delta)) \mid \Delta \in D(S), x\Delta\text{-grounded}\}$.

In the next section we will be using dependence trees to prove two central results. The first, and most important, is that if S is sound, then $S_* = S^*$. From this it follows that the domains of the S-rank function R_S and S-level function L_S are identical. Our second result is that for all x in their common domain, $R_S(x) = L_S(x)$.

7. DEPENDENCE GROUNDING = INHERITANCE GROUNDING AND RANK = LEVEL

PROPOSITION 12. Let S be a sound subset of U. Then $S_* = S^*$.

Proof: [$\supseteq$] This is immediate from Proposition 7. [$\subseteq$] Suppose $x \in S_*$. Then x is Δ-grounded for some $\Delta \in D(S)$. We prove by three induction that every object with an occurrence in $\mathcal{T}(x, \Delta)$ is in S^*, from which it follows that $x \in S^*$. If y^j is a terminal node then by Definition 8(c) y bears Δ to nothing, whence y is in the ground over S and Proposition 6 tells us that $y \in S \cup J(\emptyset)$. If $y \in S$, then since by definition of S^* $S \subseteq S^*$, $y \in S^*$. If $y \in J(\emptyset)$, then since $J(\emptyset) \subseteq J(S) \subseteq S^*$, $y \in S^*$. Now suppose y^j is not terminal; then $y \notin S$, since all members of S bear Δ to nothing. Either $\mathbb{S}(y) = \emptyset$ or $\mathbb{S}(y) \neq \emptyset$. The former is impossible because by Definition 5(γ) y would then bear Δ to itself, contradicting the Δ-groundedness of x. So $\mathbb{S}(y) \neq \emptyset$. Let M be the set of nodes lying directly below y^j. By Definition 8(c), M comprises occurrences of all and only members of some set Z sufficient for y. By hypothesis of induction, $Z \subseteq S^*$. Let α be the supremum of the S-levels of Z's members, so that $Z \subseteq J^\alpha(S)$. Then $y \in J(Z) \subseteq J^{\alpha+1}(S) \subseteq S^*$, whence $y \in S^*$ as claimed.

PROPOSITION 13. Let S be sound, and suppose $x \in S_*$. Then $R_S(x) = L_S(x)$.

Proof: [$R \geqslant L$] Let Δ be any member of $D(S)$ such that x is Δ-grounded. It is straightforward to show by tree induction that for each node y^j of $\mathcal{T}(x, \Delta)$, $R^{\mathcal{T}}(y^j) \geqslant L_S(y)$. Letting y^j be x^0 gives us $R^{\mathcal{T}}(x^0) \geqslant L_S(x)$ which by Definition 10 implies $R(\mathcal{T}(x, \Delta)) \geqslant L_S(x)$. Since Δ was arbitrarily chosen from among the members of $D(S)$ grounding x, we see by Definition 11 that $R_S(x) \geqslant L_S(x)$.

[$R \leqslant L$] Consider the S-dependence relation Δ_S defined in Section 5. x is

Δ_S-grounded by Propositions 7 and 12. It is straightforward to show by tree induction that for all y^j in $\mathscr{T}(x, \Delta_S)$, $R^{\mathscr{T}}(y^j) \leqslant L_S(y)$. In particular, $R^{\mathscr{T}}(x^0) \leqslant L_S(x)$, whence by Definition 10 $R(\mathscr{T}(x, \Delta_S)) \leqslant L_S(x)$. Since by Definition 11 $R_S(x) \leqslant R(\mathscr{T}(x, \Delta_S))$, this shows that $R_S(x) \leqslant L_S(x)$.

8. APPLICATION: TRUTH[10]

Let L be an ordinary first-order language with a distinguished predicate T, for truth. We may as well take negation, conjunction, and existential quantification as the fundamental logic constants, defining the others by means of them in the usual way. An ordered pair $M = \langle D, I \rangle$ is called an **underlying classical model** of L if D is a set containing (at least) all sentences of L and I is a function with the following properties.[11] First, I's domain is the set of names and predicates of L, i.e., the set of L's non-logical constants. Second, if c is a name of L, then $I(c) \in D$. Third, for all x in D there is a name c of L such that $I(c) = x$.[12] Fourth, if P is an n-place predicate of L (other than T), then $I(P) = (I^t(P), I^f(P))$, where $I^t(P)$ and $I^f(P)$ are disjoint and jointly exhaustive subsets of D^n. Finally, $I(T) = (I^t(T), I^f(T)) = (\emptyset, \emptyset)$.[13]

An ordered pair $M = \langle D, I \rangle$ is called a **T-partial model** of L if it satisfies all of the above conditions except, possibly, the last, in place of which we impose the weaker requirement that $I^t(T)$ and $I^f(T)$ are disjoint subsets of the collection of L's sentences. Clearly every underlying classical model of L is a T-partial model of L. If M_1 and M_2 are T-partial models of L, then we say that M_2 **T-extends** M_1, written $M_1 \leqslant M_2$, if: $D_1 = D_2$, I_1 and I_2 agree on everything other than T, and both $I_1^t(T) \subseteqq I_2^t(T)$ and $I_1^f(T) \subseteqq I_2^f(T)$.

A **partial valuation** of L is simply a partial function from the set of L's sentences into $\{t, f\}$. We follow the usual practice of treating functions, and partial functions, as sets of ordered pairs, so that a partial valuation of L may also be viewed as a subset of S of $U = \{\langle \phi, v \rangle / \phi$ a sentence of L, $v = t$ or $v = f\}$ – hereafter the set of **facts** – with the property that for no ϕ are both $\langle \phi, t \rangle$ and $\langle \phi, f \rangle$ in S. The set of these partial valuations teams up with the set of facts introduced just above to form the P and the U of our inductive space $\langle U, P, J \rangle$.

Now we have our partial models of L and our partial valuations of L. The need of the moment is clearly a way of getting from each of the former to one of the latter. A **valuation scheme** V is a function from T-partial models

M of L to partial valuations $V(M)$ of L. The only condition on V is that it be **monotonic** in the sense that $V(M) \subseteq V(M')$ whenever $M \leqslant M'$; for it takes the monotonicity of V, in this sense, to guarantee the monotonicity, in the sense of Section 2, of our soon to be defined jump operator J. The most widely employed monotonic valuation schemes are van Fraassen's supervaluational schemes and the strong and weak schemes of Kleene. For the sake of definiteness we will settle in advance on Kleene's strong scheme, although the bulk of what follows will go through on any monotonic scheme.

Strong Kleene Scheme:

(a) If P is an n-placed predicate of L (other than T) and $c_1c_2 \ldots c_n$ are names of L, then $Pc_1c_2 \ldots c_n$ is true (false) if $(I(c_1), I(c_2), \ldots, I(c_n)) \in I^t(P)$ (resp. $I^f(P)$).

(b) If $I(c)$ is the sentence ϕ of L, then $T(c)$ is true (false) if $\phi \in I^t(T)$ (resp. $I^f(T)$). If $I(c)$ isn't a sentence of L, then $T(c)$ is true (false) if $T(c) \in I^t(T)$ (resp. $I^f(T)$).[14]

(c) $-\phi$ is true (false) if ϕ is false (true).

(d) $\phi \,\&\, \psi$ is true if both ϕ and ψ are true, and false if either ϕ or ψ is false.

(e) $(\exists x)\phi$ is true if $\phi(c)$ is true for any name c of L, and false if $\phi(c)$ is false for all names c of L.

(f) Nothing is true or false except by virtue of clauses (a)–(e).

Let's fasten our attention now on just those T-partial models of L which T-extend some fixed underlying classical model M of L; M will function as our formal surrogate for "the way the world is", minus facts about the truth-values of sentences. Evidently these T-partial models differ from one another in nothing more than the way they interpret T, so that a typical one $M' = \langle D, I' \rangle$ is uniquely determined by the ordered pair $(I'^t(T), I'^f(T))$. Kripke refers to these ordered pairs as "partial sets", and it is as a function from partial sets to partial sets that the jump operator makes its appearance in his presentation. We depart slightly from Kripke by characterizing the T-extensions M' of M not in terms of the partial sets $(I'^t(T), I'^f(T))$ but the "ordinary" sets $S = \{\langle \phi, t \rangle / \phi \in I'^t(T)\} \cup \{\langle \phi, f \rangle / \phi \in I'^f(T)\}$, and avail ourselves of the easier generalizability that comes of construing the jump as an operator on sets. Every T-extension M' of M emerges with a canonical representation 'M_S', where $\{\phi / \langle\phi, t\rangle \in S\}$ is the extension and $\{\phi / \langle\phi, f\rangle \in S\}$ the antiextension assigned T by M_S. The Kripke jump of S will be the set of all $\langle\phi, t\rangle$ such that ϕ is true in M_S and $\langle\phi, f\rangle$ such that ϕ is false in M_S. And since this is nothing but the partial valuation induced by M_S, we have the defining equation $J_M(S) = V(M_S)$ for the jump operator J_M corresponding to the choice of M as underlying classical model of L.

That the union of an increasing sequence of partial valuations is again a partial valuation is obvious, as is the membership of $\emptyset$ in the set of partial valuations; the monotonicity of J_M is immediate from the monotonicity of V, so by the opening remarks of Section 2 our inductive space $\langle U, P, J_M\rangle$ is complete. Following the order of development of Sections 2–7, we turn now to the characterization of sufficiency sets. The following are easy consequences of the provisions of the strong Kleene scheme outlined above.

Sufficiency Sets:

(a) Let P be an n-place predicate (other than T), let $c_1 c_2 \ldots c_n$ be names, and let v be t or f. Then $\mathbb{S}(\langle Pc_1c_2 \ldots c_n, v\rangle)$ is P or $\emptyset$ according to whether $(I(c_1), I(c_2), \ldots, I(c_n)) \in I^v(P)$ or not.

(b) If $I(c)$ is the sentence ϕ, then $\mathbb{S}(\langle T(c), v\rangle)$ is $\{S \mid \langle\phi, v\rangle \in S\}$, whether v is t or f. If $I(c)$ isn't a sentence, then $\mathbb{S}(\langle T(c), v\rangle) = \{S \mid \langle T(c), v\rangle \in S\}$.

(c) $\mathbb{S}(\langle -\phi, v\rangle) = \mathbb{S}(\langle\phi, \bar{v}\rangle)$, where $\bar{v}$ is t if v is f and vice versa.

(d) $\mathbb{S}(\langle\phi \,\&\, \psi, t\rangle) = \mathbb{S}(\langle\phi, t\rangle) \cap \mathbb{S}(\langle\psi, t\rangle)$, and $\mathbb{S}(\langle\phi \,\&\, \psi, f\rangle) = \mathbb{S}(\langle\phi, f\rangle) \cup \mathbb{S}(\langle\psi, f\rangle)$.

(e) $\mathbb{S}(\langle(\exists x)\phi(x), t\rangle) = \bigcup_c \mathbb{S}(\langle\phi(c), t\rangle)$, and $\mathbb{S}(\langle(\exists x)\phi(x), f\rangle) = \bigcap_c \mathbb{S}(\langle\phi(c), f\rangle)$, where in both cases c ranges over all names of L.

Let's look now at some simple examples of dependence. If A is the sentence "The sentence 'Snow is white' is true and the sentence 'The sentence 'Grass is green' is true' is true" – in the familiar notation $T(\ulcorner Ws \urcorner) \,\&\, T(\ulcorner T(\ulcorner Gg \urcorner) \urcorner)$ – which partial valuations are sufficient for $\langle A, t\rangle$? By the above, $\mathbb{S}(\langle A, t\rangle) = \mathbb{S}(\langle T(\ulcorner Ws \urcorner), t\rangle) \cap \mathbb{S}(\langle T(\ulcorner T(\ulcorner Gg \urcorner) \urcorner), t\rangle) = \{S \mid \langle Ws, t\rangle \in S\} \cap \{S \mid \langle T(\ulcorner Gg \urcorner), t\rangle \in S\} = \{S \mid \langle Ws, t\rangle$ and $\langle T(\ulcorner Gg \urcorner), t\rangle \in S\}$. Since $\mathbb{S}(\langle T(\ulcorner Gg \urcorner), t\rangle)$ is in turn $\{S \mid \langle Gg, t\rangle \in S\}$, we see by Definitions 5 and 8 that the following is a possible dependence tree for $\langle A, t\rangle$:

$$
\begin{array}{ccc}
& \langle T(\ulcorner Ws \urcorner) \,\&\, T(\ulcorner T(\ulcorner Gg \urcorner)\urcorner), t\rangle & \\
\diagup & & \diagdown \\
\langle Ws, t\rangle & & \langle T(\ulcorner Gg \urcorner), t\rangle \\
& & | \\
& & \langle Gg, t\rangle
\end{array}
$$

Note that since $\mathbb{S}(\langle Ws, t\rangle) = \mathbb{S}(\langle Gg, t\rangle) = P$, and the empty set is in P, $\langle Ws, t\rangle$ and $\langle Gg, t\rangle$ need not bear Δ to anything, as depicted above.

Next, let L be the Liar sentence, i.e., the sentence $-Tn$, where $I(n) = -Tn$; note that L in effect asserts its own falsehood. By the above, $\mathbb{S}(\langle -Tn, t\rangle) = \mathbb{S}(\langle Tn, f\rangle) = \{S \mid \langle -Tn, f\rangle \in S\}$, and likewise $\mathbb{S}(\langle -Tn, f\rangle) = \{S \mid \langle -Tn, t\rangle \in S\}$. By Definition 5, every $\emptyset$-dependence relation relates $\langle L, t\rangle$ to $\langle L, f\rangle$ and vice versa, so that the "minimal" dependence tree for $\langle L, t\rangle$ looks like this:

$$
\begin{array}{c}
\langle -Tn, t\rangle \\
| \\
\langle -Tn, f\rangle \\
| \\
\langle -Tn, t\rangle \\
| \\
\langle -Tn, f\rangle \\
\vdots
\end{array}
$$

As a final example, suppose Dean says $D =$ 'Something Nixon says is false' (in symbols $(\exists x)(Nx \,\&\, -Tx)$); then which partial valuations are sufficient for the truth of Dean's utterance? By the above, $\mathbb{S}(\langle(\exists x)(Nx \,\&\, -Tx), t\rangle) = \cup_c \mathbb{S}(\langle Nc \,\&\, -Tc, t\rangle) = \cup_c [\mathbb{S}(\langle Nc, t\rangle) \cap \mathbb{S}(\langle -Tc, t\rangle)]$. Since $\mathbb{S}(\langle Nc, t\rangle)$ is $\emptyset$ or P according to whether $I(c)$ is in $I(N)$ or not, this is the same thing as $\cup_n \mathbb{S}(\langle -Tn, t\rangle)$, where n ranges over names of Nixon's utterances. Finally, $\cup_n \mathbb{S}(\langle -Tn, t\rangle) = \cup_n \mathbb{S}(\langle Tn, f\rangle) = \cup_n \{S \mid \langle I(n), f\rangle \in S\} = \{S \mid \exists \phi \in I(N)(\langle \phi, f\rangle \in S)\}$. In words, then, the partial valuations sufficient for the truth of Dean's statement are those assigning the value f to at least one of Nixon's statements. Among these are many assigning f to hundreds and thousands of the things Nixon says, but so far as the generation of $\langle D, t\rangle$ is concerned, these are *redundant*; the dependence approach handles the generality of Dean's statement with a proliferation of dependence *relations*, not a proliferation of *dependencies*. So if, for example, Nixon is known to have stated that snow is black, this is enough for Dean's statement to be true; or, in the terms of our formalization, the partial valuation $\{\langle Bs, f\rangle\}$ is entirely sufficient for $\langle D, t\rangle$. Since a dependence relation has to relate $\langle D, t\rangle$ to exactly the members of some partial valuation sufficient for it, we see that the following is a dependence tree for $\langle D, t\rangle$:

$$\begin{array}{c} \langle(\exists x)(Nx \,\&\, -Tx), t\rangle \\ | \\ \langle Bs, f\rangle \end{array}$$

Now the fact that trees 1 and 3 terminate, while tree 2 does not, suggests that the termination of a tree with topmost node $\langle\phi, t\rangle$ ought to have something to do with the truth of ϕ. And this is so. Suppose we refer to the $\emptyset$-dependence relations – that is, those whose ground consists of just the ordinary "non-semantic" facts – as the **dependence relations** simpliciter.[15] Then Proposition 12 is enough to show that a sentence ϕ is **grounded-true** (in Kripke's sense) if for some dependence relation Δ the fact $\langle\phi, t\rangle$ is Δ-grounded, or equivalently, $\mathscr{T}(\langle\phi, t\rangle, \Delta)$ is a terminating tree. (And similarly, of course, for **grounded-false**.) Thus semantical groundedness, on this approach, turns out to be just what our intuitions have always told us: *unpackability into ordinary non-semantic facts.*

9. APPLICATION: LEVEL-SEEKING

An important, but somewhat obscure, aspect of Kripke's Theory is the ability of sentences to "seek their own level" within his hierarchy. Although there is obviously a lot more to be said about it, two points can be made.

First, the level of a sentence is not determined by – and in particular is not bound to rise above – the levels of the sentences which it is *about*, but only the levels of the sentences on which it *depends.*[16] The resulting flexibility contributes crucially to the Theory's capacity for discrimination of "securely benign" from "potentially vicious" self-reference. To see how, suppose that Dean truly says that some of Nixon's statements are false, and that Nixon's statements comprise exactly 'Snow is black' and 'Some things Dean says are in English'. Then *nothing threatens* (as it *would* with Russell's Ramified Theory of Types) to require Dean's statement about Nixon and Nixon's about Dean to have levels each higher than the other; and this is because while Dean's statement is *about* both of Nixon's, it only *depends* on Nixon's statement about snow, and similarly Nixon's statement about Dean does not *depend* on Dean's statement about Nixon, or indeed on any of Dean's statements.

Suppose we define a dependence relation as **strict** if none of its proper subrelations are dependence relations. Evidently the strict relations are those relating facts only to those facts on which they depend, as against (among others) those which they are merely about. It is easy to show both that every dependence relation has a strict subrelation, and that if Δ is a subrelation of Δ', then $R(\mathscr{T}(\langle\phi, v\rangle, \Delta)) \leqslant R(\mathscr{T}(\langle\phi, v\rangle, \Delta'))$ (provided, of course, that both of these trees terminate). Since the rank of a fact is the minimum of the ranks of all the terminating dependence trees it generates, we see that it is only the *strict* dependence relations which figure in the determination of rank.

But there is more to level-seeking than this. Of the several levels associated with the various ways in which a sentence *could* have acquired its truth-value from the sentences on which it depends, it is not constrained to abide at the highest, but is free to sink to the lowest. In rough terms, each sentence is given the chance to get its truth-value in the easiest possible way. So if Dean truly says that some of Nixon's statements are false, and Nixon's statements are the two falsehoods 'Snow is black' and 'Everything Dean says is false', then *nothing threatens* (as with *both* Russell's Ramified Theory of Types and Tarski's Theory of Truth) to require Dean's statement about Nixon and Nixon's about Dean to have levels each higher than the other; Dean's statement acquires its truth from Nixon's comment on snow, and its level is not bound to respect the circumstances that it could have acquired its truth in an entirely different way.

Now the "several levels associated with the various ways in which a sentence could have acquired its truth-value" are precisely the ranks of the various terminating dependence trees generated by the sentence in question under the operation of strict dependence relations. Thus we may say that if ϕ is grounded-true, the levels "accessible" to ϕ are the ranks of the trees $\mathscr{T}(\langle\phi, t\rangle, \Delta)$, where Δ ranges over all strict dependence relations with respect to which $\langle\phi, t\rangle$ is grounded. The "finding" by ϕ of its level can now be understood as ϕ's selection of the smallest level accessible to it.

10. APPLICATION: PARADOXICALITY

What makes a sentence paradoxical? According to our best intuitions, this: when we unravel and chase down the sentence's truth- *or* falsity-conditions, we are led into something absurd. And "absurd" here can only mean one of two things: either we are led to call a true (false) sentence false (true) (as when, for example, we choose to deny that Epimenides was really a Cretan), or we are led to maintain of a sentence that it is both true and false (as when we concede Epimenides's nationality and elect to wrestle with the resulting self-dependence of his utterance).[17]

The machinery developed above provides an ideal framework in which to make these intuitions precise. Suppose we call a fact $\langle\phi, t\rangle$ ($\langle\phi, f\rangle$) **unfaithful** if ϕ is grounded-false (resp., grounded-true), and suppose we call facts which agree in their first entry and conflict in their second **opposite**. A sentence ϕ is said to be **paradoxical** if for every dependence relation Δ and truth-value v, *either* there are Δ-paths extending from $\langle\phi, v\rangle$ to *unfaithful* facts, *or* there are Δ-paths extending from $\langle\phi, v\rangle$ to *opposite* facts.[18]

The preceding can be considered a dependence-style characterization of the paradoxes based on a dependence-oriented intuition about the nature of paradox. An interesting *inheritance*-oriented intuition about paradox is this: a sentence is paradoxical if it not only lacks a truth-value but cannot be consistently *supplied* with one. This is the intuition on which Kripke seems to be relying in this inheritance-style characterization of the paradoxical sentences: a sentence ϕ is **paradoxical** in Kripke's sense if neither $\langle\phi, t\rangle$ nor $\langle\phi, f\rangle$ is in the fixed-point Q^* generated by *any* sound partial valuation Q. It is a striking confirmation of the appropriateness of both definitions that the classes of paradoxes they respectively determine *exactly coincide*. Calling a sentence **dependence-paradoxical** if it's paradoxical in my sense, and **inheritance-paradoxical** if it's paradoxical in Kripke's, we have the following:

CLAIM. A sentence is dependence-paradoxical if and only if it is inheritance-paradoxical.

Proof: [$\Rightarrow$] Suppose ϕ is not inheritance-paradoxical; then for some truth-value v and sound partial valuation Q, $\langle\phi, v\rangle \in Q^*$. Since by Proposition 12 $Q^* = Q_*$, there must be a $\Delta \in D(Q)$ such that $\langle\phi, v\rangle$ is Δ-grounded. We extend Δ to $\bar{\Delta}$ as follows: $\bar{\Delta} = \Delta \cup \{(x, y) | x, y \in Q\}$. It is straightforward to show that $\bar{\Delta}$ satisfies Definition 5 with $S = \emptyset$, and therefore that $\bar{\Delta} \in D(\emptyset)$. I claim that no $\bar{\Delta}$-paths extend from $\langle\phi, v\rangle$ to unfaithful or opposite facts. It suffices to show that there are no unfaithful or opposite facts with occurrences in $\mathscr{T}(\langle\phi, v\rangle, \bar{\Delta})$. Clearly the facts occurring in $\mathscr{T}(\langle\phi, v\rangle, \bar{\Delta})$ are just the facts occurring in $\mathscr{T}(\langle\phi, v\rangle, \Delta)$ plus the members of Q. But all these facts are in Q_*, the former because they're Δ-grounded, and the latter because $Q \subseteq Q_*$. Since $Q_* = Q^*$, it suffices to show that Q^* contains no opposite or unfaithful facts. That Q^* contains no opposite facts follows from its being a partial valuation. If Q^* contained an unfaithful fact $\langle \theta, w \rangle$, then $\langle \theta, \bar{w} \rangle \in \emptyset^* \subseteq Q^*$, so Q^* would contain opposite facts after all. Contradiction. Therefore Q^* contains no unfaithful facts. It follows that ϕ is not dependence-paradoxical.

[$\Leftarrow$] Suppose ϕ is not dependence-paradoxical. Then there's a truth-value v and a $\Delta \in D(\emptyset)$ such that $\mathscr{T}(\langle \phi, v \rangle, \Delta)$ contains no occurrences of opposite or unfaithful facts. Let Q be the set of facts with occurrences in $\mathscr{T}$. Since $\mathscr{T}$ contains no occurrences of opposite facts, Q is a partial valuation. I claim that Q is sound, i.e., $Q \subseteq J(Q)$. Since Q contains no unfaithful facts, we see from Section 8's characterization of $\mathbb{S}(x)$ that every fact in Q has a non-empty sufficiency set. Given any x in Q, let x^α be an arbitrary occurrence of x in $\mathscr{T}$. If Q_x is the set of facts with occurrences lying directly below x^α, then we know from Definitions 5 and 9, together with the fact that $\mathbb{S}(x)$ is non-empty, that Q_x is sufficient for x. So for each x in Q, $x \in J(Q_x) \subseteq J(Q)$, whence $Q \subseteq J(Q)$ and Q is sound. But then $\langle \phi, v \rangle \in Q$ implies $\langle \phi, v \rangle \in Q^*$, and it follows that ϕ is not inheritance-paradoxical.

Results like this make it fairly plain that the inheritance and dependence approaches to semantic grounding are alternative and complementary lines of sight on the same target. Certainly neither has an obvious claim to intuitive or formal primacy over the other. The inheritance approach is perhaps a little easier to think through, but this I attribute to the way in which it manages to wrap up all of the complications of the grounding situation into a single tidy package: the jump-operator. On the same metaphor, the dependence approach can be seen as a systematic undoing of all the ribbons and laying of the contents out on the table. The result is bound to take up more room; but for this there might be adequate compensation in the ease with which we can locate among the neatly spread items the sources of some of our semantical intuitions.

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NOTES

* This paper presupposes familiarity with Kripke's Theory of Truth. See Kripke's (1975) 'Outline of a theory of truth'. This paper owes a great deal to the encouragement

and specific assistance of Hans Herzberger. Points of particular indebtedness are acknowledged as they arise.

[1] Ever since Mirimanoff (1917).

[2] There is, of course, a third possibility, namely that semanticists are not convinced that the theory is an attractive one. There is no question that it has serious drawbacks, but it is still far and away the best we've got, and I will be concerned with developing the theory, rather than criticizing it, in this paper.

[3] Examples of formally analogous inheritance-style characterizations can be drawn almost at random from the plentiful stock of inductive definitions currently in use in most parts of mathematics. To name a few: (i) the definition of the grounded sets in terms of the $R(\alpha)$'s; (ii) the definition of the constructible sets in terms of the $L(\alpha)$'s; (iii) the definition of the primitive recursive functions as those in the closure of the set containing only the constant, successor, and projection functions under the operations of composition and primitive recursion; (iv) the definition of the logical consequences of T as the sentences in the closure of T under the appropriate rules of inference; (v) the standard definition of a semigroup generated by given basis elements; and (vi) the standard definition of the Borel sets.

[4] Examples of dependence-style characterizations are a little harder to come by. But consider (i) the definition of the grounded sets as those not heading any infinite descending ϵ-chains, and (ii) the definition of the logical consequences of T as the sentences with proofs (from T).

[5] See Pascal (1655), Behmann (1937), Langford (1947), Ryle (1951), Ayer (1953), Skinner (1959), Martin (1967) and (1968), Herzberger (1970), Kneale (1971) and (1972), and Mackie (1973).

[6] Almost none of the material in Section 2 is new; a lot of it is adapted from Moschovakis (1974). The notion of soundness herein employed I learned from H. Herzberger, with whom I believe it is, in the present abstract context, original.

[7] The notion of jump-entailment is due to H. Herzberger, who also made valuable suggestions about its application in an abstract setting.

[8] Clause (γ) takes the form it does for largely technical reasons. The idea is this: if x is not in S, and there are no sets sufficient for x, then we want x to head infinite Δ-paths for every Δ in $D(S)$. Insisting that x bear Δ to itself is just the simplest way of achieving this. Intuitively, we can think of such x's as having been reduced, in the absence of any independent means of support, to a futile dependence on themselves.

[9] Clause (ii) is there to ensure that the order type of every Δ-path is less than or equal to ω.

[10] The literature contains another dependence-style treatment of Kripke-grounding, namely Lawrence Davis's 'An Alternate Formulation of Kripke's Theory of Truth'. The advantages of the present approach are three. First, it is based on general considerations about the relation between inheritance and dependence, and therefore generalizes easily to non-semantical contexts. Second, Davis's approach applies only to the kind of Kripke-grounding you get when you use the strong Kleene valuation scheme. It can be adapted without much trouble to Kleene's weak scheme, but it is not apparent how Davis could handle the supervaluational schemes, or, indeed, any monotonic valuation scheme other than Kleene's. Third, Davis remarks that his inability to effect a dependence-style recovery of Kripke-level constitutes a "difference between the semantics in its K (i.e., Kripke) and Downward (i.e. Davis) forms which seems worth noting" (Davis, 1979, p. 295). But see Sections 6, 7 for a dependence-style definition of 'rank' and a proof that rank and level are coincident notions.

[11] Note that underlying classical models are not, strictly speaking, *classical* models; they are rather the classical "parts" of the T-partial models defined below.
[12] This is because we will be adopting a substitutional interpretation of the quantifiers. This is done purely for convenience, in particular because we want to avoid a lengthy detour through satisfaction.
[13] Note that I do not follow Kripke in putting all non-sentences into T's antiextension.
[14] Clause (b)'s second part has been added for purely technical reasons; it makes for a simplification in our treatment of paradox in Section 10 below.
[15] This is an important abbreviation to keep in mind.
[16] In what follows I will use 'depends' in the sense of 'potentially depends'. More exactly, a fact e depends on a fact e' if e' is a necessary part of some set S sufficient for e (i.e., if for some S, S is sufficient for e but S-$\{e'\}$ is not). In this sense, $\langle$'There are false sentences', $t\rangle$ depends on $\langle$'Snow is black', $f\rangle$, as well as on $\langle$'It's true that $0 = 1$', $f\rangle$. The sense in question must be sharply distinguished from the *stronger* sense of 'depends' according to which e depends on e' if e' is a necessary part of *every* set S sufficient for e. Since there are many intuitively "dependent" facts with no strong dependencies – $\langle$'There are false sentences', $t\rangle$ is one – the weaker notion seems to be the more useful of the two. The notion of sentence-dependence is parasitic on that of fact-dependence.
[17] I assume here that Epimenides's utterance is to the effect that all Cretan utterances are false, and that all *other* Cretan utterances *are* false.
[18] It is important to keep in mind here (and in what follows) that the *dependence relations* have been defined to be the members of $D(\emptyset)$. See the end of Section 8.

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