Explanation, Extrapolation, and Existence

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Mark Colyvan (2010) raises two problems for 'easy road' nominalism about mathematical objects. The first is that a theory's mathematical commitments may run too deep to permit the extraction of nominalistic content. Taking the math out is, or could be, like taking the hobbits out of *Lord of the Rings*. I agree with the 'could be', but not (or not yet) the 'is'. A notion of logical subtraction is developed that supports the possibility, questioned by Colyvan, of bracketing a theory's mathematical aspects to obtain, as remainder, what it says 'mathematics aside'. The other problem concerns explanation. Several grades of mathematical involvement in physical explanation are distinguished, by analogy with Quine's three grades of modal involvement. The first two grades plausibly obtain, but they do not require mathematical objects. The third grade is likelier to require mathematical objects. But it is not clear from Colyvan's example that the third grade really obtains.

1. How math helps

Quine bases his case for platonism on the role that mathematics plays in natural science. Nominalists do not deny this role. They argue that math can play it without being true — or, if this is different, without being TRUE in a sense that requires the existence of mathematical objects. It's a two-stage argument, in most cases. The nominalist begins by identifying a certain something as 'math's contribution'. She then tries to show that a theory's truth-value is beside the point when it comes to contributing in the specified way.

But there is a suppressed premise here, to the effect that math does not *also* contribute in other ways, which *do* require it to be true. This is where Colyvan sees a problem (Colyvan 2010). Nominalists have been overlooking, or underestimating, the role math plays in (i) fixing physical contents, and (ii) explaining physical outcomes. They make the first mistake when they assume that a theory's nominalistic content can be obtained by skimming the mathematical froth off the top. They make the second mistake when they neglect the possibility of physical phenomena obtaining for partly mathematical reasons.

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2. Theft vs honest Field work

Neither mistake can be pinned on Field, Colyvan thinks. He specifies *T*'s nominalistic content *directly*, by way of an explicitly formulated nominalistic sub-theory of *T*; it is not the result of some obscure de-mathification process. And far from overlooking the possibility of math playing a descriptive or explanatory role, he considers and rejects it.[1] Colyvan's concern is with post-Fieldian nominalists who, as he sees it, are oblivious to, or cavalier about, the full range of what math can do.

Nominalists have not exactly *ignored* the possibility of multiple contributions. Math is a theoretical juice extractor for Field, but expressive and explanatory contributions come in for consideration as well (as just noted). He looks at the use of mathematical methods in logic, seeking to account for it, too, 'without assuming the truth of the mathematics that is being applied' (Field 1984, p. 95). Math's role in (re)configuring information structure — the term used by linguists for the ways a given truth-conditional content can be organized and displayed — is stressed by Hofweber and Yablo.[2] A kind of 'clutch plate' function is pointed out by Maddy. Researchers may not know what in a theory is supposed to be true, as opposed to playing a placeholder or bookkeeping role. Is that four-dimensional manifold a bona fide spatiotemporal entity, or just a way of representing distances? If the first, which of its properties (density, continuity) are supposed to really belong to this entity, and which are posited for some other reason? Framing the theory mathematically enables us to postpone this decision until we feel ready to make it.[3]

[1] '[T]here may be observations that we want to formulate that we don't see how to formulate without reference to numbers, or there may be explanations that we want to state that we can't see how to state without reference to numbers … if such circumstances do arise, then we will have to genuinely accept numerical theory if we are not to reduce our ability to formulate our observations or our explanations. … *My own view is that such circumstances do not arise*' (Field 1989, pp. 161–2, italics added). This interesting passage anticipates the main themes of Colyvan's paper.

[2] Hofweber 2005, Yablo 1998, 2005. See also Gaifman et al. 1990. As Colyvan says on p. 286, I am more of an anti-platonist than a nominalist (Yablo 2009).

[3] See Maddy 1997. She cites a remark of Einstein's: 'Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place' (quoted at p. 151). Godfrey-Smith makes a related observation in his work on model-based science (Godfrey-Smith 2009).

3. Two roads

That nominalists have not been blind to these issues does not mean that they have noticed, or properly appreciated, the contributions highlighted by Colyvan. Is he right that the easy road bends back toward the hard one, when we bear those contributions in mind? Let us first remind ourselves of what the roads are.

The hard road is the one mapped out by Field in *Science Without Numbers* (Field 1980). It begins by splitting a scientific theory *T* into nominalistic and mathematical subtheories *N* and *M*. One shows that *T* is conservative over *N*, in the sense that any nominalistic consequence of *T* is provable already from *N*.[4] 'Conservative over' being a relation on theories, we are to think of *N* as linguistic in nature — as made up of sentences — rather than as the proposition ***N*** — a set of models or worlds, perhaps — that the sentences collectively express.

The easy roads have been multiplying, and I would not want to attempt a full survey. I will focus on Melia's *way of the weasel* and Yablo's *hermeneutic trail*. Both employ a strategy of 'saying less with more'. Both have us quasi-asserting *T* to really assert *T*'s nominalistic content ***N***. Both construe ***N*** on the model of 'what *T* says about the physical world'. Both claim it as an advantage of this approach that it does not require us to formulate a nominalistic theory *N* over which *T* is conservative. One simply repeats the original theory *T*, on the understanding that one is vouching only for what it says where the physical world is concerned.

This claimed advantage is what Colyvan calls into question. The easy approach 'cannot succeed without presupposing the success of Field's nominalisation program — or something like it' (Colyvan 2010, p. 287).

Colyvan does not clearly explain what 'something like it' means here. On one reading, he is claiming that easy-roaders, no less than Field, must find for each viable physical theory *T* a nominalistic subtheory *N* that captures its nominalistic content. This seems too strong. Colyvan concedes the possibility of essential metaphors in science. This is tantamount, in the present context, to accepting that there can be *T*s that are not amenable to the kind of nominalistic paraphrase that is provided by *N*.[5]

[4] This is rough; see Field 1980, pp. 10–14 for details.

[5] 'Many believe that metaphors without literal translation can carry descriptive content. For the purposes of the present debate, I will grant this and instead press the point in terms of explanation. Indeed, I think a better case can be made in terms of explanation since it is very

Is it that easy-roaders are committed to *T* having a nominalistic content in the first place? This seems too weak. Nominalistic content is not an unwanted commitment of the view under discussion; it is an essential part of that view. The easy road is easy because ***N*** is not held hostage to the existence of a theory *N* that expresses it. A commitment that is not unwanted might still be objectionable, of course. My point is just that a commitment to nominalistic content is not a commitment to 'something like' Field's program. The point of contention between two views should not be billed as an overlooked point of agreement between them.

I said that the easy road is meant to take us straight to ***N***, with no detour through a nominalistic theory *N*. I did not say how this is supposed to work. I used to conceive ***N*** as made up of all and only worlds whose physical character is such as to make *T* pretence-worthy in a certain metaphorical make-believe game.[6] But that way of putting it is apt to be misunderstood. 'Pretence' sounds like *making* as if you believe, when you do not — and all I meant by it is *being* as if you believe, without regard to whether the belief is true. The construction in those earlier papers can, however, be replayed in the key of weaseling. Tactical, assertorically inert, presuppositions take the place of mathematical make-believe.[7] The problem of specifying ***N*** becomes an instance of the general problem of scaling a stronger hypothesis back to a weaker one.[8]

One could try a *negative* sort of strategy, in which we take unwanted assumptions back at the end. This is the way of the weasel, advocated by Melia in (Melia 2000).[9] But a positive approach is possible too. Rather than backing off of what we did *not* want to talk about, we might try homing in on the subject matter we *did* want to talk about, and asserting only the part of *T* that concerned *that*.[10] On the negative approach, ***N*** is the *remainder* when we subtract from *T* its

common among scientific realists, at least, to take explanations to be ontologically committing' (Colyvan 2010, p. 301).

[6] In the papers discussed by Colyvan.

[7] 'Assertoric inertia' is Larry Horn's term (Horn 2002, Horn 2011). See also Abbott 2000.

[8] Yablo 2006, Yablo MS. For a linguistic perspective, see Von Fintel 1993 and Gajewski MS.

[9] Though see footnote 9 of that paper.

[10] I follow Lewis 1988 in treating subject matters **m** as equivalence relations on worlds. Worlds are equivalent where **the number of stars** is concerned iff they have equally many stars. Worlds are equivalent where **the physical** is concerned iff they are physically

mathematical component M. On the positive approach, $\boldsymbol{N}$ is the part of T that concerns **the physical**.[11]

Now we come to Colyvan's objections. One, directed at Melia, concerns T's nominalistic content. Why think that $\boldsymbol{N}$ is well-defined, absent a theory N such that $\boldsymbol{N}$ is the set of N-worlds? Colyvan's other objection, directed at me, is this: even if $\boldsymbol{N}$ were well-defined, it could not necessarily do the same work as T, for T's mathematical bits may be explanatorily essential.

4. Nominalistic content

Start with weaseling's ability to deliver a determinate content.[12] Colyvan asks why there should be anything worthwhile left, when T's mathematical implications are taken back. Not only do we lack a well-defined general notion of logical remainder, we know of cases where there is just no such thing as A's surplus content vis-à-vis a weaker hypothesis B. Suppose A is *Lord of the Rings* and B states the existence of hobbits. One looks in vain for an R that is suitably independent of B and such that A is equivalent to B&R. Or, try to subtract *The tomato is red* from *The tomato is scarlet* (Searle and Körner 1959, Woods 1967, Kraemer 1986). Or, try to subtract *Someone is thirsty* from *I'm thirsty*, or *I swam* from *I swam three laps*.

Wittgenstein suggests an example in this neighborhood. What is left, if we subtract from a man's raising his arm the fact of that arm going up (Wittgenstein 1953, para. 621; Jaeger 1973; Hudson 1975)? The usual answer is: his *trying* to raise it. But someone's trying to raise an arm that does in fact go up does not imply that they raised it — the arm might have been blown up by some lucky gust of wind — and it would seem a natural condition on $A-B$ that it should combine with B to imply A. Is $A-B$ his trying *effectively* to raise it? Not that either, for his effectively trying to raise his arm entails all by itself that the arm goes up. $A-B$ should not imply A all by itself, but only when combined with B.

Implications cannot be relied on in general to be detachable from their impliers. If a theory's nominalistic content is T stripped of its mathematical implications, then nominalistic contents cannot be

indiscernible. S is true about $\mathbf{m}$ if it can be made true, *period*, without changing how matters stand with respect to $\mathbf{m}$.

[11] The part of S about subject matter $\mathbf{m}$ is a proposition true in w just if S is true about $\mathbf{m}$ in w.

[12] This section is a 'friend of the court' brief. Melia may see things differently.

assumed to exist. Why would *M* be any more detachable from *T* than a tomato's redness is detachable from its particular shade of red, or the existence of hobbits is detachable from *Lord of the Rings*? These are good questions. But it is not clear the easy-roader has to answer them. There might be other ways than subtraction to scale a content back (section 4.1). Then too, it is not clear she *cannot* answer them. Subtraction is not quite as chancy an affair as we have been led to believe (section 4.2).

4.1 The high road

The easy road divides into sub-roads, corresponding to the two modes of cutting back distinguished earlier. One way to cut *A* back (the low road) is to strip away some unwanted implication *B*. Colyvan has raised doubts about this approach. But he does not consider the high-road strategy of carving out the part of *A* that concerns some pertinent subject matter **m**.

Could this, too, be ill-defined? Lewis comes very close to defining it. A hypothesis is *wholly about* **m**, according to Lewis, if its truth-value never differs between **m**-equivalent worlds. The property of being wholly about **m** is closed under conjunction, so if we let *A*/**m** be the conjunction of those of *A*'s consequences that are wholly about **m**, *A*/**m** will be wholly about **m** as well. This makes *A*/**m** a natural candidate for the role of 'what *A* says about **m**'.

A has definite implications for **m** just to the extent that *A*/**m** is evaluable in arbitrarily given worlds. Evaluation proceeds as follows: *A*/**m** is false in *w* if *A*'s falsity there is guaranteed already by *w*'s **m**-condition. (As *The number of goblins is greater than 0* is false in our world, whether there are numbers or not, thanks to the non-existence of goblins.) It is true in *w* if *A* is true there as far as *w*'s **m**-condition is concerned. (*The number of goblins is 0* is false, if it is, not for goblin-ish reasons, but because there are no numbers.) *A*/**m** is true (false) in the worlds that can (cannot) be *A*-ified with no change in how matters stand **m**-wise. Let *T* be Newtonian mechanics. The part of *T* that concerns **the physical** is the proposition *T*/**p** that is true in *w* if (i) Newtonian mechanics is true outright in *w*, or (ii) it is true there but for the absence of mathematical objects. The test, again, is whether, to turn *w* into a world where *T* is true outright, one needs to tamper with *w*'s physical condition.[13]

[13] I first encountered the idea of 'what a theory says about so and so' in Rosen 1992. Here is how he puts it in (Rosen 2001): '(1) and (2) [may] differ in what they say simpliciter, [but]

Another Lewisian notion may be helpful here. Two subject matters are *orthogonal*, he says, when the state of things with respect to one puts no constraints on the state of things where the other is concerned. **The number of cats** is orthogonal to **the number of dogs** since the size of the cat population puts no obstacles in the way of there being so and so many dogs, or vice versa. The nominalist assumes that **how matters stand physically** is orthogonal to **whether mathematical objects exist**. Take any world you like — its physical condition neither demands nor precludes the existence of numbers and sets.[14]

This might be thought to conflict with the modally inflexible, necessary-or-impossible, character of pure abstracta. But, insistence on this special character has been weakening of late. Quineans in particular should think of numbers (the way they do of electrons) as existing just where they pull their weight.[15] Even if numbers *are* necessary (impossible), that does not mean they are demanded (precluded) by how matters stand physically. **The physical** on the face of it no more demands the existence of functions than Socrates demands the existence of {Socrates} (Fine 1994, Dorr 2010). It is enough for us if nominalistic worlds are 'relatively' possible — possible where *w*'s physical condition is concerned.

The metaphysical issue of whether physical circumstances demand mathematical objects is to be distinguished from the representational issue of what it takes to *state* those physical circumstances. Numbers and functions might indeed be indispensable for this purpose. But so what? There might be no way of charting the colour relations among paint chips, except with the Munsell colour system. But the relations do not themselves *depend* on Munsell. To argue from colour relations to the existence of a suitable representation scheme would be absurd.

agree in what they say about the concrete world considered in itself. Let's say the concrete core of a world w is the largest wholly concrete part of w: the aggregate of all of the concrete objects that exist in w. What have in common is simply this: The concrete core of every world at which (1) is true is an exact intrinsic duplicate of the core of some world at which (2) is true, and vice versa', p. 75.

[14] From orthogonality it follows that any nominalistic hypothesis implied by T is implied already by T's nominalistic content. This might be considered a kind of conservativeness (Yablo 2002, section V).

[15] Field 1993; Hellman 1989; Hale and Wright 1996; Tennant 1997; Colyvan 2000, 2003; Rosen 2006.

And yet people do seem to want to argue this way, when it comes to mathematical representation. Steiner imagines a platonist thinking as follows: 'We cannot say what the world would be like without numbers, because describing any thinkable experience (except for utter emptiness) presupposes their existence' (Steiner 1978, 19-20; the formulation is Morgenbesser's). This will seem to bear on the metaphysical issue only if one slides (as Steiner does not) from 'we cannot say-without-numbers what a physically complex world would be like' to 'we cannot say what a physically complex world-without-numbers would be like'. Berkeley's whole philosophy is based on this kind of slide. To argue from *we cannot imagine-without-numbers a complex world* to *we cannot imagine a complex world lacking in numbers* is like going from *we cannot imagine a tree non-perceptually* to *we cannot imagine unperceived trees.*[16]

4.2 The low road

Now let me say a word on behalf of the low road — the one that has us 'taking things back at the end'. I agree with Colyvan that it is not always passable. But it is not always impassable, either. Logical subtraction *sometimes* yields a well-defined remainder, surely. *Snow is cold and white – Snow is cold = Snow is white*, I assume. For a generalization to be lawlike is what remains of its being a law, when we bracket whether the generalization is true. Triangles are similar if they are congruent, except they need not be the same size.

Blanket scepticism about logical remainders is no more reasonable than blanket acceptance. The sceptic is right, though, that it is not obvious how to tell the good and bad cases apart. We need a way of distinguishing

(i) *B*s which are perfectly *inextricable* from *A*

(ii) *B*s which are perfectly *extricable* from *A*

(iii) *B*s which are *somewhat* extricable from *A*

[16] I am not trying to respond here to the 'heavy-duty platonist' who thinks 'the truth of *T* at *w* depends on the obtaining there of some fundamental relations between concrete and mathematical entities which do not supervene on the totality of facts either about the concrete realm or about the narrowly mathematical relations' that *T* appeals to (Dorr 2010, p. 8). See Dorr for an enlightening discussion of this and related matters.

Perfect *in*extricability is, I take it, the case where $A-B$ is defined only on B-worlds. It is clear enough which *red* things have what-scarlet-adds-to-red: the scarlet ones. The question of which *non*-red things exemplify scarlet's incremental content — which of them fail to be scarlet only because they fail to be red — makes little sense.

B is perfectly *extricable* from A if $A-B$ is equally evaluable, and on the same basis, whether B is true or false. Same-sizedness is perfectly extricable from congruence, because figures of different sizes can be assessed for similarity just as easily as figures of the same size.

Partial extricability is the case where $A-B$ is evaluable in some $\neg B$-worlds but not others. Suppose Bert is dead. Then *Bert raises his arm* – *It goes up* is false. He falls short of raising his arm for reasons well in excess of the fact that his arm does not go up. If Bert is alive, however, and attempts unsuccessfully to raise his arm, then *Bert raises his arm* – *His arm goes up* is most likely unevaluable. Attempting to raise one's arm is not enough to make up the difference between raising it and its going up, for a reason already mentioned; the attempt does not combine with the arm going up to imply that Bert raises his arm. The next section tries to cash this 'make up the difference' metaphor out. A necessary condition has already been noted: C makes up the difference between B and A only if it implies the material conditional $B \rightarrow A$. *I tried to raise my arm* falls short because it does not imply *My arm went up* $\rightarrow$ *I raised it*.

5. Value added

Extricability turns on this question: how is $A-B$ to be evaluated in worlds where B is false? It is a technical question, which calls for a technical answer. We begin with a theory of 'incremental truth' and 'incremental falsity'. A rule is then provided for evaluating $A-B$ on the basis of A's incremental truth-value with respect to B.

Consider the relation between $p \& q$ and q, in a world where p is false. There is a clear sense in which $p \& q$ *adds* falsity to q — it commits a further offence against truth beyond any committed already by q. Similarly $p \& q$ adds truth to q if p is true — $p \& q$ is true where, and insofar as, it reaches beyond q.

Value-addedness makes intuitive sense, but we need a definition. The easy part is as follows: if B is true in w, then A adds truth or falsity according to whether A is itself true in w or false there. What if B is false in w? A adds truth to B in w if $B \rightarrow A$ has there a certain kind of truthmaker — what I will call a *targeted* truthmaker. A targeted truthmaker for $B \rightarrow A$ is a fact that (as far as possible) rules out the combination of B-true with A-false *as such*, that is,

(i) without ruling B out (= implying $\neg B$), and

(ii) without ruling A in (= implying A)

Now, (ii) is implicit in the very idea of a truthmaker. **T** cannot be considered 'that in virtue of which' S is true if it is overloaded with extra detail — detail in whose absence we would still have a condition implying S. Call that the *proportionality* requirement on truthmakers. A proportional truthmaker for $B \rightarrow A$ must take the fullest possible advantage of B; **T** can imply A, only if there is no advantage there to be taken. That makes (ii) more or less automatic, and we focus on (i). What (i) tells us is that a targeted truthmaker for $B \rightarrow A$ ($B \rightarrow \neg A$) is one that does not rule B out. **T** must be B-compatible, or, as I will put it, B-friendly.

(1) A adds truth to B in w iff
$B \rightarrow A$ has in w a B-friendly truthmaker
A adds falsity to B in w iff
$B \rightarrow \neg A$ has in w a B-friendly truthmaker[17]

The official reason why $p \& q$ adds truth to q in the case where p is true is this: $q \rightarrow p \& q$ has a q-friendly truthmaker in the fact that p. $p \& q$ adds falsity, when p is false, because $q \rightarrow \neg(p \& q)$ has a q-friendly truthmaker in the fact that $\neg p$.

(2) $A - B$ is true in w iff A adds truth to B in w (and no falsity)
$A - B$ is false in w iff A adds falsity to B in w (and no truth)

I said above that B is more or less extricable from A according to how often $A - B$ is evaluable in worlds where B is false. How often that occurs depends on how often A adds 'just truth', or 'just falsity', to B in such worlds. How often it does *that* depends on how often one, but not the other, of $B \rightarrow A$, $B \rightarrow \neg A$ has a B-friendly truthmaker in w.[18]

[17] $B \rightarrow \neg A$ is equivalent to $\neg A$, since A implies B.

[18] 'One and not the other', because an A that adds truth to B is not automatically debarred from adding falsity. (1) *Snow is hot or black* $\rightarrow$ *Snow is hot* has a hot-or-black-compatible truthmaker in the fact that snow is white. *Snow is hot or black* $\rightarrow$ *Snow is not hot* has one in the fact that snow is cold. (2) I own three birds, imagine. Alice and Bert are cockatoos, Cleo

Whether triangles are congruent-if-the-same-size turns entirely on their shapes — on whether, for instance, both are equilateral or only one is. But now, equilaterality-facts, and shape-facts more generally, obtain just as easily in worlds where the size facts are different. This is why *They are the same size* is so highly extricable from *They are congruent*.

What about *Tom is scarlet* and *Tom is red* (Tom is the aforementioned tomato)? The latter is extricable just to the extent that *Tom is red* → *Tom is scarlet* (*Tom is red* → *Tom is not scarlet*) is true in Tom-is-red-worlds for reasons that obtain just as easily when Tom is not red. As is familiar, there is no (natural, non-disjunctive) property independent of redness that combines with it to yield the property of being scarlet. The reason Tom is scarlet-if-red, in red-worlds, is that Tom is scarlet; the reason it's unscarlet-if-red, in red-worlds, is that it's (say) crimson. These are the truthmakers. They cannot obtain in worlds where Tom is not red; so its being red is highly *inextricable* from its being scarlet.

That leaves *I raised my arm* and *My arm went up*. *It went up* is extricable just to the extent that *It went up* → *I raised it* (*It went up* → *I did not raise it*) is true in arm-up worlds for reasons that can obtain when my arm is not up. What might those reasons be? *Trying* to raise my arm has the virtue of holding also in arm-down worlds, but it does not suffice for *It went up* → *I raised it* (since, again, I could have tried ineffectively at the same time as the wind blew my arm up). Trying *effectively* has the virtue of sufficing for the conditional, but it does not hold in any arm-down worlds. Truthmakers for *My arm went up* → *I did not raise it* can hold, however, in arm-down worlds. Paralysis does not require my arm to be up, as a thing's property of being scarlet requires it to be red. *My arm goes up* is thus *partly* extricable from my raising it.[19]

is a parakeet. *Both my birds are cockatoos* adds truth to *I have two birds* because of Alice and Bert; it adds falsity because of Cleo.

[19] The remainder is *false* in some arm-down worlds, but not, as far as I can see, true in any such worlds. An example of partial extricability where the remainder can also be true in $\neg B$-worlds might be this. Plantinga defines warrant for believing that p as what knowing that p adds to the fact that p (Plantinga 1993). Consider my belief that the world is not about to end. It is warranted *even if I am wrong*, and the world is (as a matter of pure quantum mechanical happenstance) about to end. The belief falls short of knowledge, I am suggesting, just in that one respect. A fact obtains that implies the following: *I am right* → *I know*.

6. The (in)extricability of mathematics

The rate of star formation has been exponentially decreasing for many billions of years. Let T be a definite hypothesis along these lines, say: the number of new stars in the nth millennium after redshift 2 (about ten billion years ago) is proportional to $k/2^n$ (for some suitable k). Writing $\#(n)$ for the number of new stars in the nth millennium, the hypothesis is that

T: $\#(n) = k/2^n$, for some suitable integer k

Its mathematical component is something like

M: Numbers exist with all the expected properties

To determine how extricable M is from T, we ask how common it is, in the numberless part of logical space, for T to add just truth, or just falsity, to M. T adds just truth, in a numberless world, if each millennium sees in twice as many new stars as the next. This ensures a number-friendly truthmaker for $M \rightarrow T$, and precludes one for $M \rightarrow \neg T$, which is what it means for T to add just truth. A millennium introducing *more* than twice as many new stars as its successor (or *fewer* than twice as many) ensures a number-friendly truthmaker for $M \rightarrow \neg T$ and precludes one for $M \rightarrow T$. T adds just falsity in such worlds. $T - M$ is evaluable, then, in worlds where each millennium sees in twice as many as, over twice as many as, or fewer than twice as many stars, as the one that follows. 'Twice as many, over twice as many, fewer than twice as many'—these exhaust the possibilities. M is thus highly extricable from T, leaving behind a well-defined remainder.

Redacting a theory's mathematical aspects is not, in super-simple cases anyway, like writing the hobbits out of *Lord of the Rings*. Sophisticated theories may be thought different in this respect. What if T is quantum mechanics, and M says there are infinite dimensional vector spaces (Hilbert spaces) with the expected properties. It is hard to think what $T - M$ could be. Quantum mechanical states are not only *played* by vectors, it seems, there is nothing for them to *be* but vectors. We are unable to frame any positive notion of what T asks of a world, leaving aside the Hilbert spaces.

This would be a problem, if we were taking a top-down approach, evaluating $T - M$ in w on the basis of some prior idea of what it says. But our approach is pointwise and bottom-up. What $T - M$ 'says'—the proposition it expresses—is given by its changing truth-values as we vary w. These truth-values are as specified in (2): the proposition is true in worlds where T adds (only) truth to M, and false in worlds where T adds (only) falsity to M. Doubts about the extricability of M from T have thus got to be doubts about the workability of such a procedure in nominalistic worlds. Running this through the definition of adding truth, they are doubts that $M \rightarrow T$ can have a non-trivial truthmaker in such worlds.[20] $M \rightarrow T$ can, on this view, have a non-trivial truthmaker only in platonistic worlds.

The sceptic has no objection, I take it, to facts holding in Hilbert-space-containing worlds which necessitate T. What she cannot make sense of is facts holding in non-Hilbert–space-containing worlds which combine with the assumption of Hilbert spaces to necessitate T. This puzzles me. What is the difference supposed to be between (i) T's actual truth-value in a w which contains Hilbert spaces, and (ii) T's hypothetical truth-value in a physically indiscernible w' that is assumed to contain them? I appreciate that w' may not be conceivable in purely physical terms. But that is not required; w' is conceived as a world which is physically just like a world w in which T is true and has nothing else going on in it. We are given that T is true in w. Somehow, though, we are unable to evaluate T in a world w' whose differences with w have been imagined away.

I suspect that the sceptic's objection arises earlier, with the idea of physically indiscernible worlds only one of which has mathematical objects. I agree that a non-mathematical T-world is in *some* very good sense beyond our comprehension. But, we must be careful not to confuse 'we cannot say-without-mathematical-objects what w would be like' with 'we cannot say (even with mathematical objects) what a non-mathematical w would be like'. Running these two together is the Berkeley fallacy again. Material objects are presented in imagination in a sensible garb; that does not mean they can't exist unperceived (Williams 1976, Peacocke 1985). Angels and devils do not have to be really there, in order for behaviour to be describable as angelic or diabolical. The sceptic maintains that it is different with mathematical objects. Hilbert spaces must really exist in worlds that are not imaginable except in terms of Hilbert spaces. This may be true in the end. But the argument for it should not be Berkeley's argument.

[20] Unless $M \rightarrow \neg T$ does as well — a qualification that will be taken as understood. A trivial truthmaker for $M \rightarrow T$ is one that is inconsistent with M and hence also T.

7. Mathematical explanation

Ptolemy sought to account for retrograde motion in terms of crystalline spheres and epicycles. One of the reasons his account proved to be worthless is that crystalline spheres and epicycles do not exist. A state of affairs has got to obtain, it seems, before it can account for anything. The explanation given today of retrograde motion is different; it relies on the mathematics of angular velocity. But that doesn't mean we can ignore the constraint that sunk Ptolemy. The mathematics involved must be true, the argument goes, to be explanatory. If, as the nominalist thinks, it is not true, then the explanation we were taught of retrograde motion is worthless too. That it does not *seem* to be worthless suggests the nominalist has made a mistake somewhere.

7.1 Grades of Mathematical Involvement

Quine started out doubtful of abstract ontology, as he was doubtful about 'real' (non-verbal, de re, Aristotelian) necessity. He reversed himself on ontology, but held his ground on modality. His strategy there was to distinguish three grades of modal involvement (Quine 1966). The first two grades, involving sentential predicates and operators, are quasi-legitimate, for Quine, but they fail to establish real necessity. The third, involving non-trivial quantification into modal contexts, would establish real necessity, were it not illegitimate.

I propose to distinguish, in a similar way, three grades of mathematical involvement in physical explanation. It will sharpen the comparison to force our explanations into a covering-law format: outcome E is explained as arising out of circumstances C by way of a generalization G that links them.[21]

Math helps *descriptively* to the extent that we need it to specify C or E, or to formulate the generalization. E might be the sinusoidal oscillation of a stressed spring. G might be Hooke's law, which gives the restorative force as a function of how far the spring is stretched or compressed.

Math helps *structurally* if it lets us run the explanation at the right level of generality.[22] To give the inevitable example: What is it about a

[21] This is simplistic, but that is in a way the point. Explanatory utility does not argue for platonism unless the math is pulling its weight. If we don't know how to judge this even in simple cases, it will be hard to run the argument in complex ones.

[22] Math's structural contributions are examined in Pérez-Carballo MS, with reference especially to the Könisberg bridge puzzle.

square peg that allows it to slip into a round hole (Putnam 1995, Garfinkel 1981)? The peg's microphysical make-up involves too much unneeded detail; it would still have fit, had it been made of copper. The peg fits if and because the sides are less than $\sqrt{2}$ times as long as the radius of the hole.

Math helps *substantively*, if it provides the covering generalization *G*. Honeycomb divides into six-sided cells, it is said, because the hexagon is the shape that tiles the plane most efficiently, providing the most coverage with the least perimeter. Why has no one on a round-trip tour through Königsberg ever managed to cross each of the town's bridges exactly once? It has parts reachable by any of three bridges, which gives the associated graph a vertex of odd degree, which was shown by Euler to make a no-doubling-back tour impossible.[23]

That is the hierarchy; now we try our hand at applying it. Here is a silly example to begin with. Certain numbers of tiles are never seen on the floors of rectangular rooms. One finds rooms with 18 tiles but not 19, 36 but not 37, 70 but not 71.[24] The outcome in need of explanation is that certain bunches of tiles, depending somehow on their cardinality, are not the right sort to cover rectangular floors. To say it precisely, a rectangular floor cannot be covered with the *X*s (*X* ranging over bunches of tiles) if the number of *X*s is 19, or 37, or 71, or etc. This is first grade involvement; we are using numbers to pick out the phenomenon that puzzles us.

The second grade is reached when we realize that the unsuitable bunches of tiles have something in common; they are prime in number. This makes sense, on reflection, since a floor that is m tiles wide and n tiles long (m, n>1) will require $m \times n$ tiles in all. A prime number of tiles will not do, because prime numbers do not factorize in the way required. That is the explanatorily relevant feature and it is mathematical in nature, which is what second-grade involvement requires. (Is there third-grade involvement? It might seem not, since the relevant generalization — prime numbers cannot be non-trivially factorized — is just the definition of 'prime'. But let's not bother ourselves about that. The primes could equally have been defined as the numbers that divide $a \times b$ only if they divide a or b. Non-factorizability then becomes a theorem.)

[23] The honeycomb and bridge examples are from Lyon and Colyvan 2008 and Pincock 2007.

[24] Tiles are one-foot squares, assume; floors have integral dimensions; 'rooms' are at least two feet wide; and so on.

Colyvan's example is more interesting. The Kirkwood gaps are gaps in the asteroid belt between Mars and Jupiter.

> The explanation for the existence and location of these gaps is mathematical and involves the eigenvalues of the local region of the solar system (including Jupiter). The basic idea is that the system has certain resonances and as a consequence some orbits are unstable. Any object initially heading into such an orbit, as a result of regular close encounters with other bodies (most notably Jupiter), will be dragged off to an orbit on either side of its initial orbit. An eigenanalysis delivers a mathematical explanation of both the existence and location of these unstable orbits.(Colyvan 2010, p. 302)

There is first-grade involvement here, because the unstable orbits have periods a certain fraction of Jupiter's; the fractions are identified mathematically as the eigenvalues of an operator. There is second-grade involvement too, because the fractional relation is explanatorily relevant. An asteroid that circles the sun three times, for each revolution of Jupiter's, will *thereby* be drawn into repeated interactions with Jupiter of a type to eventually pull it off course.

First-grade involvement does not make much of a case for platonism. Once it is granted that 'metaphors without literal translation can carry descriptive content' (p. 301), the nominalist is home free; she will say that the weight of the explanation is borne by the nominalistic descriptive content, not the platonistic packaging.[25] Colyvan's focus is, at any rate, on second-grade involvement. The Kirkwood gap mathematics must be accepted as true, because a unitary physical explanation is not possible:

> Each asteroid … will have its own complicated, contingent story about the gravitational forces and collisions that [it] has experienced. Such causal explanations are … piecemeal and do not tell the whole story. [They] do not explain why no asteroid can maintain a stable orbit in the Kirkwood gaps. The explanation of this important astronomical fact is provided by the mathematics of eigenvalues (that is, basic functional analysis). (p. 303)

[25] Colyvan says the following, however: 'when a piece of language is delivering an explanation, either that piece of language must be interpreted literally or the non-literal reading of the language in question stands proxy for the real explanation. Moreover, in the latter case, the metaphor in question must clearly deliver and identify the real explanation' (Colyvan 2010, p. 301). I do not see how to reconcile this with the concession in the text. Suppose a fact **F** that explains *E* is the descriptive content of sentence *S*. Then '*E* because *S*' says that *E* is the case on account of **F**. Since it *is* the case on account of **F**, how can this fail to constitute an explanation of *E*, and whence the further demand (I assume it is a further demand) that *S* 'must clearly deliver and identify the real explanation'?

Colyvan suggests it is a problem for nominalism if what the orbits have in common, by virtue of which they are unstable, is mathematical rather than physical. He does not say why it is a problem. The nominalist rejects mathematical *ontology*, not mathematical typology. Why should she not agree that math enables the scientist to carve physical phenomena at the explanatory joints? (A lot of it was created for just that purpose.)

The Kirkwood gap might be compared in this respect to the 'Armstrong gap'—the fact that certain numbers of tiles never turn up on rectangular floors.[26] What Colyvan says about the first applies equally to the second.

> Every bunch of tiles will have its own complicated, contingent story about the forces and collisions that it has experienced. Such causal explanations are piecemeal and do not tell the whole story. They do not explain why no rectangular floor can be covered by tiles in the Armstrong gap. The explanation of that fact is provided by the mathematics of multiplication.

Tiles fall into the Armstrong gap, says our imagined platonist, when and because the number of them is prime. Numbers had better exist, lest the gap go unexplained. The answer to this is that we can explain it without numbers: tiles fall into the gap when and because there are unfactorizably many of them. Primeness comes in, not as a feature of numbers, but as a principle of unity tying together certain concrete pluralities (Rayo 2002). 'Unfactorizably many' is not wonderful English, it is true. 'An eigenvalue-y interference-making fraction of the orbital period of Jupiter' is not English at all. But the point either way is the same. Metaphysical distinctions should not be made to depend on where exactly the line falls between good, borderline, and unacceptable English.

This brings us to the hoped-for third grade of mathematical involvement, in which the explanatory *connection* is mathematical. The orbits are unoccupied, you might think, *because* the corresponding orbital periods are eigenvalues of a certain vector operator. It is clear how the argument goes from here. Suppose a mathematical generalization packs explanatory punch, or 'carries some of the explanatory load' (p. 300). Then we have to treat it as true, or we lose the explanation. If a generalization is true, and it concerns certain objects, then the objects exist.

This sounds somewhat plausible, I admit. But the nominalist is not going to give up just yet. Questions can be raised at each stage. I will

[26] Armstrong is a tile company.

present them dogmatically, not because the platonist has no hope of answering them, but to suggest what remains to be done before the argument is complete.

Does the generalization pack explanatory punch? Punch in this case would be: the Kirkwood orbits are unoccupied because a vector operator squishes or stretches certain vectors by such and such an amount. The nominalist will reject this claim. The reason those orbits are unoccupied, in her view, is that they are unstable. Algebra helps us to *identify* the unstable orbits. But the reason they are unoccupied is to do with constructive interference. Again, bees make hexagonal cells because it is an efficient use of their energy, and to that extent adaptive. Geometry helps us to see what efficient behaviour would be in this setting, but not why the behaviour occurs.[27] Graph theory helps us identify the topographical layouts permitting an Eulerian round trip tour. But what does the explaining, in any particular case, are the facts (to do with the town's physical structure) that make the graph-theoretic result applicable.

Is it the right kind of explanation? Hempel, Salmon, and others, have argued that explanations are not all of the same kind. A hypothesis is ontically explanatory if it tells us why, objectively speaking, the outcome occurs. It is epistemically explanatory if accepting it makes the outcome less surprising, or puts one in a better position to understand it.[28] Euler's result may indeed be explanatory in the epistemic sense. But so are Boyle's law, the Bohr model of the atom, and so on. These have something right about them, but they do not paint an entirely faithful picture of reality. A hypothesis need not be true to set us on the road to appropriate expectations.

Is it the right kind of truth (i.e. mathematical)? Not all mathematical theories are of the same kind, and this bears on what we are doing when we call a mathematical statement true. Philosophers tend to focus on (quasi-)categorical theories, such as arithmetic, the theory of the continuum, and set theory. There are also algebraic theories, however, like group theory, graph theory, and Boolean algebra. Truth in a categorical theory like arithmetic can be understood as truth in the theory's intended model, which is unique up to isomorphism. Euler's result is not even a candidate for truth in this sense, for the theory has no intended models; it applies to whatever satisfies the axioms. What did Euler discover, then? He discovered *logical* truths

[27] I am indebted here to Brad Skow.

[28] It is modally explanatory if it shows why no other outcome was possible.

to the effect that anything with the structural features postulated in the axioms (Königsberg, for example) has such and such other features as well.[29]

Is the right kind of truth existence-implying? The line between arithmetical truths and falsehoods is fairly clear. There is no agreement, however, on what qualifies a statement to be put on that side of the line — in particular, on what it is about the truths that makes them true. Their distinguishing feature to the if-thenist is logical; they follow from the axioms. Their distinguishing feature to some recent structuralists is modal: they hold in all *possible* ω-sequences (Hellman 1989). Arithmetical truths on another model are like laws of nature; they say how numbers are *supposed* to behave, qua numbers, whether the kind is instantiated or not.[30] The platonist may think that these alternatives rob arithmetical truths of their explanatory punch. But it is not clear why that would be. The platonist needs to tell us what the real existence of numbers adds explanation-wise to the truth about numbers as such.

One can stipulate, of course, that mathematical truth properly so-called does require the existence of mathematical objects. Nominalists are bound to reject mathematical truth *in that sense*. But that does not mean they are bound to reject (what we are forced, given the stipulation, to call) mathematical *correctness*. And indeed they do not reject it. The usual strategy is to associate each mathematical S with a non-ontologically-loaded S^* such that S is correct iff S^* is true. I have been emphasizing the version that takes S^* to be *logically* true, but that is far from the only possibility.[31]

Our stipulation has the result that grade-three mathematical involvement in physical explanation entails the existence of mathematical objects. Just for that reason, though, it will be harder to establish the existence of grade-three involvement than initially seemed. A case will have to be made that the explaining is done by S rather than some

[29] Along with empirical facts about the relevance of this sort of logical truth to real-world problems. 'What separates a person who knows a lot of mathematics from a person who knows only a little mathematics is not that the former knows many claims [of the sort] that mathematicians commonly provide proofs of … [it is] knowledge of a purely logical sort' (Field 1984, p. 545).

[30] *A body not subject to any force remains at rest or in uniform motion* does not depend for its lawfulness on objects like that existing. There is even the possibility of a law that is uninstantiated *because* it obtains. *Brakeless trains are dangerous*, was Lewis's example.

[31] For ideas about the form S^* might take, see Rayo 2002, Linnebo and Rayo 2012, Yablo 2002, Yablo 2005, Hellman 1989, Correia 2006, Azzouni 2004, and Hofweber 2005.

closely related S^*. To refute easy-road nominalism, Colyvan must show that S's explanatory role cannot be played by a substitute that focuses more on the *Sosein* of mathematical objects than their *Sein*. He ends by saying, 'The debate over platonism and nominalism would be genuinely advanced by a better understanding of explanation — especially those explanations that have mathematics playing the leading role' (p. 304). This is true, and Colyvan is right to press the issue. A better understanding is also needed of what it is for mathematics to play the lead role in an explanation.[32]

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