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ABSTRACT OBJECTS: A CASE STUDY

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1. Necessity

Not a whole lot is essential to me: my identity, my kind, my origins, consequences of these, and that is pretty much it. Of my intrinsic properties, it seems arguable that none are essential, or at least none specific enough to distinguish me from others of my kind. And, without getting into the question of whether existence is a property, it is certainly no part of my essence to exist.

I have by contrast *huge* numbers of accidental properties, both intrinsic and extrinsic. Almost any property one would ordinarily think of is a property I could have existed without.

So, if you are looking for an example of a thing whose "essence" (properties had essentially) is dwarfed by its "accense" (properties had accidentally), you couldn't do much better than me. Of course, you couldn't easily do much *worse* than me, either. Accense dwarfs essence for just about any old object you care to mention: mountain, donkey, cell phone, or what have you.

Any old *concrete* object, I mean. Abstract objects, especially *pure* abstracta like 11 and the empty set, are a different story. I do not know what the intrinsic properties of the empty set are, but odds are that they are mostly essential. Pure sets are not the kind of thing we expect to go through intrinsic change between one world and another. Likewise integers, reals, functions on these, and so on.[1]

The pattern repeats itself when we turn to relational properties. My relations to other concrete objects are almost all accidental. But the number 11's relations to other abstract objects (especially other numbers) would seem to be essential.

The most striking differences have to do with existence. Concrete objects (with the possible exception of "the world," on one construal of that phrase) are one and all contingent. But the null set and the number 11 are thought to exist in every possible world. This is *prima facie* surprising, for one normally supposes that existence is inversely related to essence: the bigger *x*'s essence, the "harder" it is for *x* to exist, and so the fewer worlds it inhabits. And yet here is a class of objects extremely well endowed in the essence department, and missing from not even a single world.

You would have to be in a coma not to wonder what is going on here. Why is it that so much about abstract objects is essential to them? What is it about numbers *et al*. that makes it so hard for them not to exist? And shouldn't objects that turn up under all possible conditions have impoverished essences as a result?

It may be that I have overstated the phenomenon. Not everyone agrees that numbers even exist, so it is certainly not agreed that they exist necessarily. There would be more agreement if we changed the hypothesis to: numbers exist necessarily provided they *can* exist, that is, unless they're impossible.[2] And still more if we made it: numbers exist necessarily provided they *do* exist. But these are nuances and details. I think it is fair to say that *everyone*, even those who opt in the end for a different view, has trouble with the idea that 11 could go missing.

So our questions are in order, construed as questions about how things intuitively seem. Why should a numberless world seem impossible (allowing that the appearance may be only *prima facie*)? Why should it seem impossible for numbers to have had different intrinsic properties, or different relational properties *vis-à-vis* other abstract objects? Why should numbers seem so modally inflexible?

2. Apriority

A second *prima facie* difference between the concrete and abstract realms is epistemological. Our knowledge of concreta is aposteriori. But our knowledge of numbers, at least, has often been considered apriori. That 3+5=8 is a fact that we *could* know on the basis of experience—of counting, say, or of being told that 3+5=8. But the same is true of most things we know apriori. It is enough for apriority that experience does not *have* to figure in our justification. And this seems true of many arithmetical claims. One can determine that 3+5=8 just by thinking about the matter.

Like the felt necessity of arithmetic, its felt apriority is puzzling and in need of explanation. It is a thesis of arithmetic that there are these things called numbers. And it is hard to see how one could be in a position to know apriori that things like that really existed.

It helps to remember the two main existence-proofs philosophers have attempted. The ontological argument tries to deduce God's existence from God's definition, or the concept of God. The knock against this has been the same ever since Kant; from the conditions a thing would have to satisfy to be X, nothing existential follows, unless you have reason to think that the conditions are in fact satisfied. Then there is Descartes's cogito. This could hardly be expected to give us much guidance about how to argue apriori for numbers. Also, the argument is not obviously apriori. You need to know that you think, and that knowledge seems based on your experience of self.[3]

I said that the ontological argument and the cogito were the two best-known existence-proofs in philosophy. Running close behind is Frege's attempted derivation of numbers themselves. If the Fregean line is right, then numbers are guaranteed by logic together with definitions. Shouldn't that be enough to make their existence apriori? Perhaps, if the logic involved were ontology-free. But Frege's logic affirms the existence of all kinds of higher-type objects.[4] (Frege would not have wanted to *call* them objects because they are not saturated; but there is little comfort in that.) The Fregean argument cannot defeat doubts about apriori existence, because it presupposes they *have* been defeated in presupposing the apriority of Fregean logic.

A different strategy for obtaining apriori knowledge of numbers goes via the "consistency-truth principle": in mathematics, a consistent theory is a true theory. If we can know apriori that theory T is consistent, and that the consistency-truth principle holds, we have apriori warrant for thinking T is true, its existential claims included.

There are a lot of things one could question in this strategy. Where do we get our knowledge of the consistency-truth principle? You may say that it follows from the fact that consistent theories have (intended) models, and that truth is judged relative to those models. But that argument assumes the truth of model theory. And apriori knowledge of model theory does not seem easier to get than apriori knowledge of arithmetic.

Even if we do somehow know the consistency-truth principle apriori, a problem remains. Not all consistent theories are on a par. Peano Arithmetic, one feels, is *true*, and other theories of the numbers (AP) are true only to the extent that they agree with PA. It doesn't help to say that PA is true of its portion of mathematical reality, while AP is true of its. That if anything only reinforces the problem, because it makes AP just as true in its own way as PA. It begins to look as though arithmetical truth can be apriori only if we downgrade the kind of truth involved. A statement is not true/false absolutely but only relative to a certain type of theory or model.

3. Absoluteness

I take it as a given that mathematical truth doesn't *feel* relative in this way. It feels as though 3+5 is just plain 8. It feels as though the power set of a set is just plain bigger than the set itself.

It could be argued that the notion of truth at work here is still at bottom a relativistic one: it is truth according to *standard math*, where a theory is standard if the mathematical community accepts and uses it.[5]

But truth-according-to-accepted-theories is a far cry from what we want, and act like we have. For now the question becomes, why is this theory standard and not that? The answer cannot be that the theory is *true*, in a way that logically coherent alternatives are not true, because there is no truth on this view but truth-according-to-accepted-theories; to explain acceptance in terms of acceptance-relative truth would be to explain it in terms of itself. I assume then that PA's acceptance will have to be traced to its greater utility or naturalness given our projects and cognitive dispositions. But this has problematic results. Why is it that 3+5=8? Because we wound up *passing* on the coherent alternative theory according to which 3+5 is not 8—and for reasons having nothing to do with truth. Neither theory is truer than the other. That, as already stated, is not at all how it feels.

Another problem is sociological. 3 plus 5 was seen to be 8 long before anyone had formulated a theory of arithmetic. How many people even today know that arithmetic is something that mathematicians have a theory of? Saul and Gloria (my non-academic parents) are not thinking that 3+5=8 is true-relative-to-the-standard-theory, because they have no idea that such a theory exists, and if apprised of it would most likely think that the theory was standard because it was true. Are they just confused? If so, then someone should pull the scales from their eyes. Someone should make them realize that the truth about numbers and sets is (like the truth about what's polite or what's stylish) relative to an unacknowledged standard, a standard that is in relevant respects quite arbitrary. I would not want to attempt it, and not only because I don't like my parents angry at me. If they would balk at the notion that there's no more to be said for standard mathematics than for a successful code of etiquette, I suspect they're probably right.

Admittedly, there are *parts* of mathematics, especially of set theory, where a relative notion of truth seems not out of place. Perhaps the most we can say about the continuum hypothesis is that in some nice-looking models it is true, while in others it is false. I admit then that the intuition of absolute truth may not extend to all cases. But even in set theory it extends pretty far. A set theory denying, say, Infinity, or Power Set, strikes us as *wrong*, even if we have yet to put our finger on where the wrongness is coming from.

Could the explanation be as simple as this? If a model doesn't satisfy Power Set, or Infinity, then we don't see it as modeling "the sets." That Infinity holds in all models of "the sets" is a trivial consequence of that linguistic determination. It's not as if there is a shortage of models which include only *finite* set-like objects. It's just that these objects are at best the pseudo-sets, and that makes them irrelevant to the correctness of Infinity taken as a description of the sets. Infinity is "true" because models that threaten to falsify it are shown the door; they are not part of the theory's intended subject matter.

Call this the *debunking* explanation of why it seems wrong to deny the standard axioms. I do not say that the debunking explanation is out of the question; it may be that ZF serves in effect as a reference-fixer for "set." But again, that is not how it feels. If someone wants to argue that Infinity is wrong—that the hereditarily finite sets are the only ones there are—our response isn't "save your breath! deny Infinity and you're changing the subject." Our response is:

"that sounds unlikely, but let's hear the argument." No doubt we will end up thinking that the Infinity-denier is wrong. The point is that what he is wrong about is *the sets*. It *has* to be, for if he is not talking about the sets, then we are not really in disagreement.

Suppose though the debunkers are right that ZF is true because it sets the standard for what counts as a set. This still doesn't quite explain our sense that ZF is correct. Why should we be so obsessed with the *sets* as opposed to the pseudo-sets defined by theory FZ? To the extent that ZF and "sets" are a pair, curiosity about why ZF seems so right is a lot like curiosity about why the sets seem so right. It doesn't matter how the questions individuate, as long as they're both in order. And so far nothing has been said to cast doubt on this. So again, why do ZF and the sets seem so right?

4. Abstractness and Necessity

Three puzzles, then: one about necessity, one about apriority, one about absoluteness. It will be easiest to start with necessity; the other two puzzles will be brought in shortly.

The necessity puzzle has to do both with essential properties and necessary existence. About the latter it may be speculated that there is something about *abstractness* that prevents a thing from popping in and out of existence as we travel from world to world.[6] It is, as Hale and Wright put it,[7] hard to think what conditions favorable for the emergence of numbers would be, and hard to think of conditions unfavorable for their emergence. It is by contrast easy to think of conditions favorable for the emergence of Mt. McKinley. The reason, one imagines, is that numbers are abstract and Mt. McKinley is not.

But, granted that numbers do not wait for conditions to be right, how does that bear on their necessity? Explanations come to an end somewhere, and when they are gone we are left with the brute facts. Why shouldn't the existence/nonexistence of numbers be a brute fact? Traditionally existence has been the paradigm of a phenomenon not always admitting of further explanation. Granted that numbers are not contingent *on* anything, one still wants to know why they should not be contingent full stop.[8]

A second possible explanation is that it is part of the *concept* of an abstract object (a "pure" abstract object, anyway) to exist necessarily if at all. An object that appeared in this world but not others would by that alone not be abstract.

Suppose that is right; an otherwise qualified object that does not persist through all worlds does not make the cut. One might still be curious about these contingent would-be abstracta. What sort of object are we talking about here? The obvious thought is that they are *exactly like real abstracta* except in the matter of necessary existence. But the obvious thought is strange, and so let us ask explicitly: Could there be shmabstract objects that are just like their abstract cousins except in failing to persist into every world?

Fiddling with an object's persistence conditions is generally considered harmless. If I want to introduce, or call attention to, a kind of entity that is just like a person except in its transworld career—it is missing (e.g.) from worlds where the corresponding person was born in Latvia—then there would seem to be nothing to stop me. If we can have shmersons alongside persons, why not shmumbers along with numbers?

You may think that there is a principled answer to this: a principled reason why abstracta cannot be "refined" so as to exist in not quite so many worlds. If so, though, then you hold the view that we started with: there is something about *abstractness* that precludes contingency. What is it? Earlier we looked at the idea that where pure abstracta like numbers are concerned, there could be no possible basis for selection of one world over another. But why should that bother us? Why should the choice of worlds not be arbitrary, with a different number-refinement for each arbitrary choice? This is only one suggestion, of course, but as far as I am aware, the route from abstractness to necessity has never been convincingly sketched.

Suppose then that abstract objects *can* be refined. There is nothing *wrong* with shmabstract objects, on this view, it is just that they should not be confused with *abstract* objects. Another set of questions now comes to the fore. Why do we attach so much importance to a concept—abstractness—that rules out contingent existence, as opposed to another—shmabstractness—that differs from the first only in being open to contingent existence? Does the salience of numbers as against shmumbers reflect no more than a random preference for one concept over another? One would like to think that more was involved.

5. Conservativeness and Necessity

So far we have been looking at "straight" explanations of arithmetical necessity: explanations that accept the phenomenon as genuine and try to say why it arises. Attention now shifts to non-straight or "subversive" explanations. Hartry Field does not think there are any numbers. So he is certainly not going to try to *validate* our intuition of necessary existence. He might however be able to *explain the intuition away*, by reinterpreting it as an intuition not of necessity but something related. He does in fact make a suggestion along these lines. Field calls a theory *conservative* if

> it is consistent with every internally consistent theory that is 'purely about the physical world' (Field 1989, 240).

Conservative theories are theories compatible with any story that might be told about how things go physically, as long as that story is consistent in itself. (I am going to skate lightly over the controversy over how best to understand "consistent" and "compatible" here. The details are not important for what follows.)

Now, one obvious way for a mathematical theory to be conservative is for it to be *necessary*. A theory that cannot help but be true is *automatically* compatible with every internally consistent physical theory.

But, although necessity guarantees conservativeness, there can be conservativeness without it. A necessary theory demands nothing; every world has what it takes to make the theory true. A conservative theory makes no demands on the *physical* world. If the theory is false, it is false not for physical reasons but because the world fails to comply in some other way. T is conservative iff for each world in which T is false, there's another, physically just like the first, in which T is true. The theory is false then only due to the absence of nonphysical objects like numbers.

You might think of the foregoing as a kind of necessity. A conservative theory T is "quasi-necessary" in the sense that *necessarily, T is satisfiable in the obtaining physical circumstances*. Here again is Field:

> mathematical realists ...have held that good mathematical theories are not only true but necessarily true; and a clear part of the content of this (the only clear part, I think) is that mathematics is conservative... .Conservativeness might loosely be thought of as 'necessary truth without the truth.' ...I think that the only clear difference between a conservative theory and a necessarily true one is that the conservative theory need not be true... .Perhaps many realists would be content to say that all they meant when they called mathematical claims necessarily true was that they were true and that the totality of them constituted a conservative theory (Field 1989, 242).

From this it seems a small step to the suggestion that the only distinctively *modal* intuition we have about mathematical objects is that the theory of those objects is conservative. So construed, the modal intuition is quite correct. And it is correct in a way that sits well with our feeling that existence is never "automatic"—that nothing has such a strong grip on reality as to be incapable of not showing up.

Is our intuition of the necessity of "3+5=8" just a (confused) intuition of quasi-necessity, that is, conservativeness?

I think it is very unlikely. Yes, every world has a physical duplicate with numbers. But one could equally go in the opposite direction: every world has a physical duplicate without them. If the permanent possibility of adding the numbers in makes for an intuition of necessity, then the permanent possibility of taking them out should make us want to call numbers impossible. And the second intuition is largely lacking. A premise that is symmetrical as regards mathematical existence cannot explain why numbers seem necessary as opposed to impossible.

A second reason why necessity is not well-modeled by conservativeness is this. Arithmetical statements strike us as *individually* necessary. We say, "this has *got* to be true," not "this considered in the context of such and such a larger theory has got to be true." But the latter is what we *should* say if our intuition is really of conservativeness. For conservativeness is a property of particular statements only seen as exemplars of a surrounding theory. A statement that is conservative in the context of one theory might change stripes in the context of another. (Imagine for instance that it is inconsistent with the other.) Nothing like that happens with necessity.

A third problem grows out of the discussion above of consistency as sufficient for truth. Suppose that two theories contradict each other. Then intuitively, they cannot both be necessary; indeed if one is necessary then the other is impossible. But theories that contradict each other *can* both be conservative.

Someone might reply that if contradictory means *syntactically* contradictory, then contradictory theories can so be necessary. All we have to do is think of them as describing different domains (different portions of the set-theoretic universe, perhaps).

That is true in a technical sense. But the phenomenon to be explained—our intuition of necessity—occurs in a context where contradictory theories are, the technical point notwithstanding experienced as incompatible. If I affirm Infinity and you deny it, we take ourselves to be disagreeing. But both of us are saying something conservative over physics.

When two statements contradict each other, they cannot both be necessarily true. Unless, of course, the truth is *relativized*: to the background theory, a certain type of model, a certain portion of mathematical reality. This takes us out of the frying pan and into another frying pan just as hot. Once we relativize, standard mathematics ceases to be *right* (full stop). And as already discussed, a lot of it *feels* right (full stop). Once again, then, our problems about apriority and necessity are pushing us toward a no less problematic relativism.

6. Figuralism

The conservativeness gambit has many virtues, not least its short way with abstract ontology. At the same time there are grounds for complaint. One would have liked an approach that made arithmetic "necessary" without making it in a correlative sense "impossible." And one would have liked an approach less friendly to relativism.

The best thing, of course, would be if we could hold onto the advantages of the Field proposal without giving up on "real" necessity, and without giving up on the intuition of absolute truth or correctness. Is this possible? I think it just may be. I can indicate the intended direction by hazarding (what may strike you as) some extremely weird analogies:

(A) *"7 is less than 11"*
"the frying pan is not as hot as the fire"
"a molehill is smaller than a mountain"
"pinpricks of conscience register less than pangs of conscience"

(B) *“7 is prime”*
“the back burner is where things are left to simmer”
“the average star has a rational number of planets”
“the real estate bug doesn’t sting, it bites”

(C) *“primes over two are not even but odd”*
“butterflies in the stomach do not sit quietly but flutter about”
“pounds of flesh are not given but taken”
“the chips on people’s shoulders never migrate to the knee”

(D) *“the number of Fs is large iff there are many Fs”*
“your marital status changes iff you get married or ...”
“your identity is secret iff no one knows who you are”
“your prospects improve iff it becomes likelier that you will succeed”

(E) *“the Fs outnumber the Gs iff #{x|Fx} > #{x|Gx}.”*
“you are more resolute ...iff you have greater resolve”
“these are more available...iff their market penetration is greater”
“he is more audacious...iff he has more gall”

(F) *“the # of Fs=the # of Gs iff there are as many Fs as Gs”*
“your whereabouts = our whereabouts iff you are where we are”
“our greatest regret = yours iff we most regret that...and so do you”
“our level of material well-being = yours iff we are equally well off”

Here are some ways in which these statements appear to be analogous. (I will focus for the time being on necessity.)

All of the statements seem, I hope, true. But their truth does not depend on what may be going on in the realm of concrete objects and their contingent properties and relations. There is no way, we feel, that 7 could fail to be less than 11. Someone who disagrees is not understanding the sentence as we do. There is no way that molehills could fail to be smaller than mountains, even if we discover a race of mutant giant moles. Someone who thinks molehills could be bigger is confused about how these expressions work.

Second, all of the statements employ *a distinctive vocabulary*—“number,” “butterflies,” “{x|Fx},” “market penetration”—a vocabulary that can also be used to talk about concrete objects and their contingent properties. One says “the number of local affiliates is growing,” “her marital status is constantly changing,” and so on.

Third, its suitability for making contingent claims about concrete reality is the vocabulary’s *reason for being*. Our interest in stomach-butterflies does not stem from curiosity about the aerodynamics of fluttering. All that matters to us is whether people *have* butterflies in the stomach on particular occasions. Our interest in 11 has less to do with its relations to 7 than with whether, say, the eggs in a carton have 11 as their number, and what that means about the carton's relation to other cartons whose eggs have a different number.

Fourth, the vocabulary's utility for this purpose *does not depend* on conceiving of its referential-looking elements as genuinely standing for anything. It doesn't depend on conceiving its referential-looking elements any other way, either. Those if any who take stomach-butterflies, greatest regrets, and numbers dead seriously derive the exact same expressive benefit from them as those who think the first group insane. And both groups derive the exact same expressive benefit as the silent majority who have never given the matter the slightest thought.

7. Necessity as Back-Propagated

I said that all of the statements strike us as necessary, but I did not offer an explanation of why. With regard to the non-mathematical statements, an explanation is quickly forthcoming.

Stomach-butterflies and the rest are *representational aids*. They are "things" that we advert to not (not at first, anyway) out of any interest in what they are like in themselves, but because of the help they give us in describing other things. Their importance lies in the way they boost the language's expressive power.

By making as if to assert that I have butterflies in my stomach, I really assert something about how I feel—something that it is difficult or inconvenient or perhaps just *boring* to put literally. The *real content* of my utterance is the real-world condition that makes it sayable that S. The real content of my utterance is that reality has feature BLAH: the feature by which it fulfills its part of the S bargain.

The reason it seems contingent that her marital status has changed is that, at the level of real content, it *is* contingent: she could have called the whole thing off. The reason it seems necessary that our prospects have improved iff it has become likelier that we will succeed is that, at the level of real content, it *is* necessary, as the two sides say the very same thing.

How does the world have to be to hold up its end of the "the number of apostles is even" bargain? How does the world have to be to make it sayable that the number of apostles is even, supposing for argument's sake that there are numbers? There have to be evenly many apostles. So, the real content of "the number of apostles is even" is that there are evenly many apostles.

That there are evenly many apostles is a hypothesis that need not have been true, and that it takes experience to confirm. At the level of real content, then, "the number of apostles is even" is epistemically and metaphysically contingent. But there might be *other* number-involving sentences whose real contents are necessary. To the extent that it is their real contents we hear these sentences as expressing, it will be natural for us to think of the sentences as necessarily true.

This explains how number-involving sentences, e.g., "the number of Fs = the number of Gs iff the Fs and Gs are equinumerous" can feel necessary, at the same time as we have trouble seeing how they *could* be necessary. Our two reactions are to different contents. The sentence feels necessary because at the level of real content it is tautologous: the Fs and Gs are equinumerous iff they are equinumerous. And tautologies really are necessary.

The reason we have trouble crediting our first response is that the sentence's literal content—that there is this object, a number, that behaves like so—is to the effect that something exists. And it is baffling how anything could cling to existence that tightly.

Why do the two contents get mooshed together in this way? A sentence's *conventional* content—what it is generally understood to say—can be hard to tell apart from its *literal* content. It takes work to remember that the literal meaning of "he's not the brightest guy in town" leaves it open that he's the second brightest. It takes work to remember that (literally) pouring your heart out to your beloved would involve considerable mess and a lengthy hospital stay, not to mention the effect on your beloved. Since there is no reason for us to do this work, it is not generally realized what the literal content in fact is.

Consider now "$7 < 11$." To most (!) people, most of the time, it means that seven somethings are fewer than eleven somethings. But the literal content is quite different. The literal content makes play with entities 7 and 11 that measure pluralities size-wise, and encode by their internal relations facts about supernumerosity. Of course, the plurality-measures 7 and 11 are no more on the speaker's mind than blood is on the mind of someone offering to pour their heart out. "$7 < 11$" is rarely used to describe numbers as such, and so one forgets that the literal content is about nothing else.

The literal contents of pure-mathematical statements are quickly recovered, once we set our minds to it. The real contents remain to be specified. I do not actually think that the real contents are always the same, so there is a considerable amount of exaggeration in what follows. But that having been said, the claim will be that arithmetic is, at the level of real content, a body of logical truths—specifically, logical truths about cardinality—while set theory consists, at the level of real content, of logical truths of a combinatorial nature.

8. Arithmetic

Numbers enable us to make claims which have as their real contents things we really believe, and would otherwise have trouble putting into words.

One can imagine introducing number-talk for this purpose in various ways, but the simplest is probably this. Imagine that we start out speaking a first-order language with variables ranging over concreta. Numerical quantifiers "$\exists_n x$ Fx" are defined in the usual recursive way.[9] Now we adopt the following rule (*S* means that it is to be supposed or imagined that S):

(N) if $\exists_n x$ Fx, then *there is a thing n = the number of Fs*.

Since (N)'s antecedent states the real-world condition under which we're to make as if the Fs have a number, F should be a predicate of concrete objects. But the reasons for assigning numbers to concrete pluralities apply just as much to pluralities of numbers (and pluralities of both together). So (N) needs to be strengthened to

(**N**) if $*\exists_n x\ Fx*$ then *there is a thing n = the number of Fs*.

This time F is a predicate of concreta and/or numbers. Because the rule works recursively in the manner of Frege, it gets us "all" the numbers even if there are only finitely many concreta. 0 is the number of non-self-identical things, and $k+1$ is the number of numbers $\leq k$.

Making as if there are numbers is a bit of a chore; why bother? Numbers are there to expedite cardinality-talk. Saying "#Fs = 5" instead of "$\exists_5 x\ Fx$" puts the numeral in a quantifiable position. And we know the expressive advantages that quantification brings. Suppose you want to get it across to your neighbor that there are more sheep in the field than cows. Pre-(**N**) this takes (or would take) an infinite disjunction: there are no cows and one sheep or there are no cows and two sheep or there is one cow and there are two sheep, and etc. Post-(**N**) we can say simply that the number of sheep, whatever it may be, exceeds the number of cows. The real content of "#sheep > #cows" is the infinite disjunction, expressed now in finite compass.[10]

This gives a sense of the real contents of *applied* arithmetical statements are; statements of *pure* arithmetic are another matter.

Take first quantifierless addition statements. What does the concrete world have to be like for it to be the case that, assuming numbers, 3+5=8? Assuming numbers is assuming that there is a number k numbering the Fs iff there are k Fs. But that is not all. One assumes that if no Fs are Gs, then the number of Fs and the number of Gs have a sum = the number of things that are either F or G. All of that granted, the real-world condition that makes it OK to suppose that 3+5 = 8 is that

$$\exists_3 x\ Fx\ \&\ \exists_5 y\ Gy\ \&\ \forall x\ \neg(Fx\&\ Gx) \rightarrow \exists_8 z\ (Fz \vee Gz).$$

This is a logical truth. Consider next quantifierless multiplication statements. What does the concrete world have to be like for it to be the case that, assuming numbers, $3 \times 5 = 15$? Well, it is part of the number story that if n = the number of F_1s = the number of F_2s = … the number of F_ms, and there is no overlap between the F_is, then m and n have a product $m \times n$ = the number of things that are F_1 or F_2 or … .F_m. With that understood, the real-world condition that entitles us to suppose that $3 \times 5 = 15$ is

$$(\exists_3 x\ F_1 x\ \&\ \ldots\&\ \exists_3 x\ F_5 x\ \&\ \neg\exists x\ (F_1 x\ \&\ F_2 x)\ \&\ldots)\rightarrow$$
$$\exists_{15} x\ (F_1 x \vee \ldots \vee F_5 x).$$

Once again, this is a logical truth. Negated addition and multiplication statements are handled similarly; the real content of $3+5 \neq 9$, for example, is that

$$\exists_3 x\, \mathrm{F}x \;\&\; \exists_5 y\, \mathrm{G}y \;\&\; \forall x\, \neg(\mathrm{F}x \& \mathrm{G}x) \rightarrow \neg \exists_9 z\, (\mathrm{F}z \vee \mathrm{G}z).$$

Of course, most arithmetical statements, and all of the "interesting" ones, have quantifiers. Can logically true real contents be found for them?

They can, if we help ourselves to a few assumptions. First, the real content of a universal (existential) generalization over numbers is given by the countable conjunction (disjunction) of the real contents of its instances. Second, conjunctions all of whose conjuncts are logically true are logically true. Third, disjunctions any of whose disjuncts are logically true are logically true. From these it follows that

The real content of any arithmetical truth is a logical truth.

Atomic and negated-atomic truths have already been discussed.[11] These give us all arithmetical truths (up to logical equivalence) when closed under four operations: (1) conjunctions of truths are true; (2) disjunctions with truths are true; (3) universal generalizations with only true instances are true; (4) existential generalizations with any true instances are true. It is not hard to check that each of the four operations preserves the property of being logically true at the level of real content. We can illustrate with case (4). Suppose that $\exists x\phi(x)$ has a true instance $\phi(n)$. By hypothesis of induction, $\phi(n)$ is logically true at the level of real content. But the real content of $\exists x\phi(x)$ is a disjunction with the real content of $\phi(n)$ as a disjunct. So the real content of $\exists x\phi(x)$ is logically true as well.

9. Set Theory

Sets are nice for the same reason as numbers. They make possible sentences whose real contents we believe, but would otherwise have trouble putting into words. One can imagine introducing set-talk for this purpose in various ways, but the simplest is probably this. "In the beginning" we speak a first-order language with quantifiers ranging over concreta. The quantifiers can be singular or plural; one can say "there is a rock such that it..." and also "there are some rocks such that they ...." Now we adopt the following rule:

(S) if there are some things *a*, *b*, *c*..., then *there is a set {*a*, *b*, *c*,...}*.

Since the antecedent here states the real-world condition under which we're to make as if *a*, *b*, *c*,... form a set, *a*, *b*, *c*,... are limited to concrete objects. But the reasons for collecting concreta into sets apply just as much to the abstract objects introduced via (S). So (S) is strengthened to

(**S**) if* there are some things *a*, *b*, *c*...*, then *there is a set {*a*, *b*, *c*,...}*.

This rule, like (**N**) in the last section, works recursively. On the first go-round we get sets of concreta. On the second go-round we get sets containing concreta and/or sets of concreta. On the third we get sets containing concreta, sets of them, and sets of *them*. And so on through all the finite ranks. Assuming that there are only finitely many concreta, our output so far is the *hereditarily finite* sets: the sets that in addition to being themselves finite have finite sets as their members, and so on until we reach the concrete objects that started us off.

What now? If we think of (**S**) as being applied at regular intervals, say once a minute, then it will take all of eternity to obtain the sets that are hereditarily finite. No time will be left to obtain anything else, for example, the first infinite number ω.

The answer to this is that we are not supposed to think of (**S**) as applied at regular intervals; we are not supposed to think of it as applied at all. (**S**) does not say that when we *establish* the pretense-worthiness of "there are these things," it *becomes* pretense-worthy that "they form a set." It says that if as a matter of fact (established or not) *there are these things,* then *there is the set of them.* If *there are the hereditarily finite sets*, then certainly *there are the von Neumann integers ($0 = \phi$, $n+1 = \{0,1,...n\}$)*. And now (**S**) tells us that *there is the set {0, 1, 2, 3,...},* in other words, *there is ω*. I believe (but will not try to prove) that similar reasoning shows we get all sets of rank α for each ordinal α. (**S**) yields in other words the full tower of sets: the full cumulative hierarchy.

Now, to say that (**S**) yields the full cumulative hierarchy might seem to suggest that (**S**) yields *a certain fixed bunch* of sets, viz. all of them. That is not the intention. There would be trouble if it were the intention, for (**S**) leaves no room for a totality of all sets. To see why, suppose for contradiction that *a, b, c,... are all the sets*. (**S**) now tells us that *all the sets form a set V*. This set V must for familiar reasons be different from a, b, c,... . So the proposed totality is not all-encompassing. (I will continue to say that (**S**) yields the full cumulative hierarchy, on the understanding that the hierarchy is not a fixed bunch of sets, since any fixed bunch you might mention leaves something out. This does not prevent a truth-definition, and it does not prevent us from saying that some sentences are true of the hierarchy and the rest false.[12])

Conjuring up all these sets is a chore; why bother? The reason for bothering with numbers had to do with *cardinality*-type logical truths. Some of these truths are infinitely complicated, but with numbers you can formulate them in a single finite sentence. Something like that is the rationale for sets as well. The difference is that sets help us to deal with *combinatorial* logical truths—truths about what you get when you combine objects in various ways.

An example will give the flavor. It is a theorem of set theory that if $x = y$, then $\{x,u\} = \{y,v\}$ iff $u=v$. What combinatorial fact if any does this theorem encode? Start with "$\{x,u\} = \{y,v\}$." Its real content is that they$_{xu}$ are them$_{yv}$ —or, to dispense with the plurals, that ($x=y$ or $x=v$) & ($u=y$ or $u=v$) & ($y=x$ or $y=v$) & ($v=x$ or $v=u$). Thus what our theorem is really saying is that

If $x=y$, then
$([(x=y \vee x=v) \wedge (u=y \vee u=v) \wedge (y=x \vee y=v) \wedge (v=x \vee v=u)]$ iff $u=v)$.

This is pretty simple as logical truths go. Even so it is not really comprehensible; I at least would have trouble explaining what it says. If truths as simple as this induce combinatorial bogglement, it should not be surprising that the set-theoretic formulations are found useful and eventually indispensable.

A second example is Cantor's Theorem. What is the logical truth here? One can express *parts* of it using the plural quantifier ∃X ("There are some things such that..."). Numerical *plural* quantifiers are defined using the standard recursive trick:

$$\exists_0 X\ \phi(X) \text{ iff } \forall X\ \neg\ \phi(X)$$
$$\exists_{n+1} X\ \phi(X) \text{ iff } \exists Y\ (\phi(Y)\ \&\ \exists_n X\ (\phi(X)\ \&\ \neg X=Y))$$

Consider now $\exists_4 X\ \forall y\ (Xy \rightarrow Fy)$. I can't give this a *very* natural paraphrase, because English does not quantify over pluralities of pluralities. But roughly the claim is that there are four ways of making a selection from the Fs.[13] This lets us express part of what Cantor's Theorem is "really saying", viz. that if there are n Fs, then there are 2^n ways of selecting just some of the Fs, as follows:

$$\exists_n x\ Fx \rightarrow \exists_{2^n} X\ \forall y\ (Xy \rightarrow Fy).$$

This is a second-order logical truth, albeit a different such truth for each value of n. But we are still a long way from capturing the Theorem's real content, because it applies to infinite pluralities as well. There is (as far as I know) no way with the given resources to handle the infinite case.[14] It all becomes rather easy, though, if we are allowed to encode the content with sets. All we need say is that every set, finite or infinite, has more *subsets* than it has *members*. ($|P(X)| = 2^{|X|} > |X|$.)

Now let me try to give a general recipe for finding real contents. It will be simplest if we limit ourselves to talk of hereditarily finite sets; the procedure I think generalizes but that remains to be checked. Take first atomic sentences, that is, sentences of the form $x=y$ and $x \in z$. A reduction function **r** is defined:

(A_1) $\mathbf{r}(x \in z)$ is

1.	$\exists y\ ((\vee_{u \in z}\ y = u)\ \&\ x = y)$	if z has members
2.	$\exists y\ (y \neq y\ \&\ x = y)$	if z is the empty set
3.	$x \in z$	if z is not a set.[15]

Note that the first line simplifies to $\vee_{y \in z}\ x=y$; that is in practice what I will take the translation to be. (The reason for the quantified version is that it extends better to the case where z is the empty set.) The third line marks the one place where $\in$ is not eliminated. If z is not a set, then it is (literally) false to say that x belongs to it, which is the result we want. The rule for identity-statements is

(A_2) $\mathbf{r}(x=y)$ is

1. $\forall u\ (u \in x \leftrightarrow u \in y)$ if x and y are sets
2. $x=y$ if either is not a set.

In the "usual" case, x and y have members, and $\forall u\ (u \in x \leftrightarrow u \in y)$ reduces to $(\wedge_{u \in x} \vee_{v \in y}\ u=v) \wedge (\wedge_{v \in y} \vee_{u \in x}\ v=u)$. If x has members and y is the null set, it reduces to $\forall u\ (\vee_{z \in x}\ u = z \leftrightarrow u \neq u)$. If both x and y are the null set, we get $\forall u\ (u \neq u \leftrightarrow u \neq u)$. Otherwise $\mathbf{r}$ leaves $x=y$ untouched. Non-atomic statements reduce to truth-functional combinations of atomic ones by the following rules:

(R_1) $\mathbf{r}(\neg\ \phi)$ is $\neg\ \mathbf{r}(\phi)$

(R_2) $\mathbf{r}(\wedge_i\ \phi_i)$ is $\wedge_i\ \mathbf{r}(\phi_i)$

(R_3) $\mathbf{r}(\vee_i\ \phi_i)$ is $\vee_i\ \mathbf{r}(\phi_i)$

(R_4) $\mathbf{r}(\forall x\ \phi(x))$ is $\wedge_{z=z}\ \mathbf{r}(\phi(z))$.

(R_5) $\mathbf{r}(\exists x\ \phi(x))$ is $\vee_{z=z}\ \mathbf{r}(\phi(z))$.

The real content of ϕ is found by repeatedly applying $\mathbf{r}$ until you reach a fixed point, that is, a statement ϕ^* such that $\mathbf{r}(\phi^*) = \phi^*$. This fixed point is a truth-functional combination of "ordinary" statements true or false for concrete (non-mathematical) reasons. These ordinary statements are to the effect that $x = y$, where x and y are concrete, or $x = y$, where one is concrete and the other is not, or $x \in z$, where z is concrete.[16]

How do we know that a fixed point will be reached? If ϕ is a generalization, the (R_i)s turn it into a truth-functional combination of atoms ψ. If ψ is an atom talking about sets, then the (A_i)s turn it into a generalization about sets of a lower rank, and/or non-sets. Now we apply the (R_i)s again. Given that ϕ contains only finitely many quantifiers, and all the sets are of finite rank, the process must eventually bottom out.[17] The question is how it bottoms out, that is, the character of the sentence ϕ^* that gives ϕ's real content.

I claim that if ϕ is a set-theoretic truth, then ϕ^* is, not quite a logical *truth*, but a logical *consequence* of basic facts about concreta: identity- and distinctness-facts, and facts to the effect that concreta have no members. To have a word for these logical consequences, let's call them *logically true over concrete combinatorics*, or for short *logically true*$_{cc}$. Three assumptions will be needed, analogous to the ones made above for arithmetic. First, the real content of a universal (existential) generalization is given by the countable conjunction (disjunction) of its instances. Second, conjunctions all of whose conjuncts are logically true$_{cc}$ are themselves logically true$_{cc}$. Third, disjunctions any of whose disjuncts are logically true$_{cc}$ are logically true$_{cc}$.

Every set-theoretic truth has a logically true$_{cc}$ real content.

The set-theoretic truths (recall that we are limiting ourselves to hereditarily finite sets) are the closure of the atomic and negated-atomic truths under four rules: (1) conjunctions of truths are true; (2) disjunctions with truths are true; (3) universal generalizations with only true instances are true; (4) existential generalizations with any true instances are true. The hard part is to show that atomic and negated-atomic truths are logically true$_{cc}$ at the level of real content. The proof is by induction on the ranks of x and y.

Basis Step

(a) If x and y are concrete, then the real content of $x = y$ is that $x = y$. This is logically true$_{cc}$ if true, because it's a consequence of itself. Its negation is logically true$_{cc}$ if true for the same reason.
(b) If x is concrete and y is a set, then $x \neq y$ is true. Its real content $x \neq y$ is logically true$_{cc}$, because a consequence of the fact that $x \neq y$.
(c) If x and y are the null set, then $x = y$ is true. Its real content $\forall u\ (u \neq u \leftrightarrow u \neq u)$ is a logical truth, hence logically true$_{cc}$.
(d) If x is a non-empty set and y is the null set, then $x \neq y$ is true. Its real content $\neg \forall u\ (\vee_{z \in x} u = z \leftrightarrow u \neq u)$ is logically true, hence logically true$_{cc}$.
(e) If y is a non-set then $x \notin y$ is true. Its real content $x \notin y$ is logically true$_{cc}$ because a consequence of itself.
(f) If y is the null set then $x \notin y$ is true. Its real content $\neg\ \exists z\ (z \neq z\ \&\ x = z)$ is logically true$_{cc}$ because logically true.

Recursion Step

(a) If and y are nonempty sets, then $\mathbf{r}(x = y)$ is $(\wedge_{u \in x} \vee_{v \in y}\ u = v) \wedge (\wedge_{v \in y} \vee_{u \in x}\ v = u)$. (a1) If it is true that $x = y$, then $\mathbf{r}(x = y)$ is a conjunction of disjunctions, each of which has a true disjunct $u = v$. By hypothesis of induction, these true disjuncts have logically true$_{cc}$ real contents. So $\mathbf{r}(x = y)$ has a logically true$_{cc}$ real content. And the real content of $\mathbf{r}(x = y)$ is also that of $x = y$. (a2) If it is true that $x \neq y$, then $\mathbf{r}(x \neq y)$ is a disjunction of conjunctions, each of which is built out of true conjuncts. By hypothesis of induction, these true conjuncts are logically true$_{cc}$ at the level of real content. So $\mathbf{r}(x \neq y)$ has a logically true$_{cc}$ real content. And the real content of $\mathbf{r}(x \neq y)$ is also that of $x \neq y$.

(b) If z is a nonempty set, then $\mathbf{r}(x \in z)$ is $\vee_{y \in z}\, x$=y. (b1) If it is true that $x \in z$, this has a true disjunct $x = y$. By hypothesis of induction, $x = y$ has a logically true$_{cc}$ real content. But then $\mathbf{r}(x \in z)$ is logically true$_{cc}$ at the level of real content, whence so is $x \in z$. (b2) If the truth is rather that $x \notin z$, then $\mathbf{r}(x \notin z)$ is a conjunction of true conjuncts. By hypothesis of induction, these conjuncts are logically true$_{cc}$ at the level of real content. So $\mathbf{r}(x \notin y)$ has a logically true$_{cc}$ real content, whence so also does $x \notin y$.

10. Summing Up

The view that is emerging takes something from Frege and something from Kant; one might call it "Kantian logicism." The view is Kantian because it sees mathematics as arising out of our representations. Numbers and sets are "there" because they are inscribed on the spectacles through which we see other things. It is logicist because the facts that we see through our numerical spectacles are facts of first-order logic.

And yet the view is in another way the opposite of Kantian. For Kant thinks necessity is imposed *by* our representations, and I am saying that necessity is imposed *on* our representations by the logical truths they encode. Another possible name then is "*anti*-Kantian logicism." I will stick with the original name, comforting myself with the notion that the "anti" in "Kantian" can be thought of as springing into semantic action when the occasion demands.

Back now to our three questions. Why does mathematics seem (metaphysically) necessary, and apriori, and absolute? The first and second of these we have answered, at least for the case of arithmetic and set theory. It seems necessary because the real contents of mathematical statements are logical truths. And logical truths really are necessary. It seems apriori because the real contents of mathematical statements are logical truths. And logical truths really are apriori.

That leaves absoluteness. It might seem enough to cite the absoluteness of logical truth; real contents are not logically true relative to this system or that, they are logically true period.

But there is an aspect of the absoluteness question that this fails to address. The absoluteness of logic does perhaps explain why individual arithmetical statements seem in a non-relative sense correct. It does not explain why Peano Arithmetic strikes us as superior to arithmetical theories that contradict it. It does not tell us why the Zermelo-Fraenkel theory of sets strikes us as superior to set theories that contradict it. For it could be that PA is not the only arithmetical theory—ZF is not the only set theory—with the property that its real content is logically true. AP and FZ could be (at the level of real content) just as logically true as PA and ZF. Let me say something about the ZF side of this problem.

If FZ has a logically true real content, it is *not* the content induced by the game sketched above: the game based on principle (**S**). (Remember, FZ proves some A such that ZF proves ¬A. Unless something has gone very wrong, A and ¬A will not come out assertible in the same game.) FZ can be "correct" only if real contents are judged relative to a *different* principle than

> if it is to be imagined that there are some things *x*, *y*, *z*,..., then it is to be imagined that there is a set of those things.

This gives us a way out of our difficulties. I said early on that you cannot accuse someone of changing the subject just because they deny some principle of ZF. *But principle (**S**) is a great deal more basic than anything found in ZF*. If someone has trouble with the idea behind (**S**)—the idea that when you have got a determinate bunch of things, you are entitled to the *set* of those things—then that person arguably *doesn't* mean the same thing by "set" as we do.[18]

Suppose we call a theory "ZF-like" if it represents the sets as forming a cumulative hierarchy. Then here is an argument that only ZF-like theories get the sets right. If FZ is not ZF-like, then by definition it does not represent sets as forming a cumulative hierarchy. But the cumulative hierarchy comes straight out of (**S**), the rule that says that if you've got the objects, you've got the set of them as well. So, whatever it is that FZ describes, it is not a system of entities emerging (**S**)-style out of their members. Emerging (**S**)-style out of your members is definitive, though, of the sets as we understand them. FZ may well get something right, but that something is not the sets.[19]

Notes

1. Although on the Frege-Russell definition of number, there is, arguably, intrinsic change. The empty set can change too, if as Lewis suggests it is definable as the sum of all concreta. But I am talking about what we *intuitively* expect, and no one would call these definitions intuitive.
2. Wright & Hale suggest in "Nominalism and the Contingency of Abstract Objects" that Field might not accept even that much. Field *does* say that numbers are conceptually contingent. But it would be hard to pin a metaphysical contingency thesis on him, for two reasons. (1) He is on record as having not much use for the notion of metaphysical necessity. (2) To the extent that he tolerates it, he understands it as conceptual entailment by contextually salient metaphysical truths. If salient truths include the fact that *everything is concrete*, then (assuming they are not concrete) numbers will come out metaphysically impossible.
3. Burge (2000) takes a different view (p. 28).
4. See Rayo and Yablo (2000) for an interpretation that (supposedly) frees the logic of these commitments.
5. See Balaguer (2001) for discussion.
6. Impure abstracta like singleton-Socrates are not thought to be necessarily existent. So really I should be talking about pure-abstractness. I'll stick to "abstract" and leave the qualification to be understood. (Thanks here to Marian David.)

7. Hale and Wright (1996).
8. Hale and Wright expect this objection, but think it can be met.
9. $\exists_0 x\ Fx =_{df} \forall x(Fx \rightarrow x \neq x)$, and $\exists_{n+1} x\ Fx =_{df} \exists y(Fy\ \&\ \exists_n x(Fx\ \&\ x \neq y))$
10. There is an analogy here with Hartry Field's views on "the reason" for having a truth-predicate, in the absence of any corresponding property.
11. Mario Gomez-Torrente pointed out that some atomic truths have not been fitted out with real contents, a fortiori not with logically true real contents. An example is $(3+2)+1 = 6$. This had me worried, until he pointed that these overlooked atomic truths were logically equivalent to non-atomic truths that hadn't been overlooked. For instance, $(3+2)+1 = 6$ is equivalent to $\exists y\ ((3+2 = y)\ \&\ (y + 1 = 6))$. A quick and dirty fix is to think of overlooked sentences as inheriting real content from their not overlooked logical equivalents. A cleaner fix would be desirable, but Mario hasn't provided one yet.
12. See the last few pages of Putnam (1967) and "Putnam Semantics" in Hellman (1989).
13. Alternatively, there are some things all of which are Fs, and some things not the same as the first things all of which are Fs, and etc. (Say there are two Fs. You can pick both of them, either taken alone, or neither of them. Note that "all the Fs" and "none of them" are treated here as limiting cases of "some of the Fs.")
14. You could do it with plural quantification over ordered pairs.
15. The idea is that '$x \in z$' describes x as one of the things satisfying the condition of membership in z. The condition for membership in $\{a, b, c, ...\}$ is $x=a \vee x=b \vee x=c \vee$ ...The condition for membership in the null set is $x \neq x$.
16. Statements of the first type are necessarily true or necessarily false, depending on whether x is indeed identical to y. Statements of the second and third types are necessarily false, since concreta cannot be sets or have members.
17. The same argument would seem to work with sets of infinite rank; there are no infinite descending chains starting from infinite ordinals either.
18. This might sound funny, given the widespread view that there are *some* things (the sets) that are too many to form a set. This widespread view is at odds with (**S**) only if it is supposed that there is some definite bunch of things including all and only the sets. If the sets are a definite bunch of things, it is very hard to understand what could be wrong with gathering them together into a further set. I agree with Putnam when he says that "no concrete model [of Zermelo set theory] could be maximal—nor any *non*concrete model either, as far as that goes. Even God could not make a model for Zermelo set theory that it would be *mathematically* impossible to extend, and no matter what 'stuff' He might use. ...it is not necessary to think of sets as one system of objects...in order to follow assertions about all sets" (1967, 21).
19. I am grateful to a number of people for criticism and advice; thanks above all to Gideon Rosen, Kit Fine, Gilbert Harman, Mario Gomez-Torrente, Marian David, Ted Sider, Paul Horwich, and Stephen Schiffer.

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